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CONTINUITY OF WEAK SOLUTIONS TO CERTAIN SINGULAR PARABOLIC EQUATIONS

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ABSTRACT

It is demonstrated that weak solutions of (1.1) in the introduction are continuous in their domain of definition. The continuity up to the boundary is also investigated.

AMS(MOS) Subject Classifications: 35K10, 35K15, 35K20, 35K65

Key Words: singular or degenerate evolution equations, free boundary, Stefan problem.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The singular parabolic equations treated here serve as a model of heat conduction in processes where a change of phase occurs, such as water-ice, solidification of alloys, melting of metals.

Usually solutions of boundary value problems associated with these equations are found in a global sense, i.e. they are defined as equivalence classes in certain Sobolev spaces. It is of interest to decide whether they may be defined pointwise and if they possess some local regularity such as continuity.

In this paper we prove that global (weak) solutions are in fact continuous. Moreover we study under what circumstances the continuity can be extended up to the boundary of the domain where the process takes place.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CONTINUITY OF WEAK SOLUTIONS
TO CERTAIN SINGULAR PARABOLIC EQUATIONS

Emmanuele Di Benedetto

1. Introduction:

In this paper we study the continuity of weak solutions of parabolic "equations," with principal part in divergence form, of the type

$$\frac{\partial}{\partial t}\beta(u) - \operatorname{div}\vec{\underset{\sim}{a}}(x,t,u,\nabla_x u) + \underset{\sim}{b}(x,t,u,\nabla_x u) \ni 0 \tag{1.1}$$

in the sense of distributions over a domain Q in $\mathbb{R}^{N+1}$.

Here $\beta(\cdot)$ represents a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ such that $0 \in \beta(0)$, $\vec{\underset{\sim}{a}}$ is a map from $\mathbb{R}^{2N+2}$ into $\mathbb{R}^N$ and $\underset{\sim}{b}$ maps $\mathbb{R}^{2N+2}$ into $\mathbb{R}^1$.

Beside their intrinsic interest, inclusions such as (1.1) arise as a model to a variety of diffusion problems. In particular they comprehend in a unifying scheme, free-boundary problems of different nature. We mention specifically problems of fast chemical reaction [5, 8, 9], diffusion in porous media [1, 3, 4, 13, 20, 27], diffusion in porous media of partially saturated gas [14, 25], problems of diffusion involving change of phase of Stefan type [1, 6, 13, 16, 18, 20, 28].

Here we deal with the case in which $\beta(\cdot)$ has a jump at the origin. More precisely we assume $\beta(\cdot)$ is given by

$$\beta(r) = \begin{cases} \beta_1(r) & r > 0 \\ [-\nu, 0] & r = 0 \\ \beta_2(r) - \nu & r < 0 \end{cases} \tag{1.2}$$

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where $\nu > 0$ is a given constant and $\beta_i(\cdot)$ $i = 1, 2$, are monotone increasing functions in their respective domain of definition, a.e. differentiable and

(1.3) $$0 < \alpha_0 \le \beta_i'(r) \le \alpha_1 \quad , \quad i = 1, 2$$

for two positive constants α_0, α_1.

We introduce some notation and make precise the meaning of solution of (1.1).

Let Ω be a bounded domain in $\mathbb{R}^N$ of boundary $\partial\Omega$ and for $0 < T < \infty$ let $\Omega_T \equiv \Omega \times (0,T]$, $\Omega(t) \equiv \Omega \times \{t\}$, $S_T = \bigcup_{0<t\le T} \partial\Omega \times \{t\}$, $\Gamma = S_T \cup \Omega(0)$.

For $q, r \ge 1$ we denote by $L_{q,r}(\Omega_T)$ the Banach space of those measurable functions mapping $\Omega_T \to \mathbb{R}$, with norm defined by

$$\|u\|^r_{q,r,\Omega_T} = \int_0^T \|u\|^r_{q,\Omega}(t)dt$$

where

$$\|u\|^q_{q,\Omega}(t) = \int_\Omega |u(x,t)|^q dx \quad .$$

When $q = r = 2$, $L_{2,2}(\Omega_T)$ coincides with the Hilbert space $L_2(\Omega_T)$ whose inner product $(\cdot,\cdot)_{2,\Omega_T}$ generates the norm $\|\cdot\|_{2,\Omega_T} \equiv \|\cdot\|_{2,2,\Omega_T}$.

Let $W_2^{1,0}(\Omega_T)$ denote the Hilbert space with inner product

$$(u,v)_{W_2^{1,0}(\Omega_T)} = (u,v)_{2,\Omega_T} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{2,\Omega_T} ,$$

while $W_2^{1,1}(\Omega_T)$ denotes the Hilbert space with inner product

$$(u,v)_{W_2^{1,1}(\Omega_T)} = (u,v)_{W_2^{1,0}(\Omega_T)} + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{2,\Omega_T} .$$

Here $\frac{\partial u}{\partial x_i}$, $\frac{\partial u}{\partial t}$ denote generalized derivatives. With $\overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ we denote the space of those elements in $W_2^{1,1}(\Omega_T)$ whose trace on $\partial\Omega \times (0,T]$ is zero.

Let $V_2^{1,0}(\Omega_T)$ denote the Banach space of functions such that the map $t \to u(\cdot,t)$ is continuous with respect to $\|\cdot\|_{2,\Omega}$, and the norm is given by

$$|u|^2_{V_2^{1,0}(\Omega_T)} = \sup_{0\le t\le T} \|u(\cdot,t)\|^2_{2,\Omega} + \|\nabla_x u\|^2_{2,\Omega_T} ,$$

where

$$\|\nabla_x u\|^2_{2,\Omega_T} = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right)_{2,\Omega_T} .$$

From (1.2) it follows that $r \to \beta(r)$ is a relation in $\mathbb{R} \times \mathbb{R}$, whose inverse $\beta^{-1}(\cdot)$ is a function.

Definition: By a weak solution of (1.1) in Ω_T we mean a function $u \in W_2^{1,1}(\Omega_T)$ defined by

$$u \equiv \beta^{-1}(w) ,$$

where w is a function defined in Ω_T such that

$$w \subset \beta(u) ,$$

the inclusion being intended in the sense of the graphs, and w and u satisfy

$$(1.4) \qquad \int_\Omega w(x,\tau)\varphi(x,\tau)dx \Big|_{t_0}^{t} + \int_{t_0}^{t} \int_\Omega \{-w(x,\tau)\frac{\partial}{\partial t}\varphi(x,\tau) +$$

$$+ \vec{\underset{\sim}{a}}(x,\tau,u,\nabla_x u) \cdot \nabla_x\varphi + \underset{\sim}{b}(x,\tau,u,\nabla_x u)\varphi\}dxd\tau = 0$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$, and all intervals $[t_0,t] \subset (0,T]$.

If $u \in V_2^{1,0}(\Omega_T)$ is solution of a boundary value problem associated with (1.1), then it satisfies (1.4), the boundary conditions being specified separately. We remark that if in (1.4) we want to allow $t_0 = 0$, then along with $u(x,0) = u_0(x)$, the selection $w_0(x) \subset \beta(u_0(x))$ must be given. A common device consists in prescribing $u_0(x) \neq 0$ a.e. in Ω so that $\beta(u_0(x))$ is unambiguously a.e. defined in Ω.

We are not concerned here with the existence of weak solutions of (1.1), for which we refer to [1, 5, 6, 16, 18, 20]. Our results are local in nature and descent only from identity (1.4), so that we need not associate (1.1) with a particular boundary value problem.

Our goal is to prove that a weak solution of (1.1) is continuous in Ω_T. For this we introduce the auxiliary function

$$v(x,t) = \beta_0(u(x,t)) \equiv \begin{cases} \beta_1(u(x,t)) & , \text{ on } [u > 0] \\ 0 & , \text{ on } [u = 0] \\ \beta_2(u(x,t)) & , \text{ on } [u < 0] \end{cases} ,$$

and set

$$w(x,t) = v(x,t) - \nu(x,t)\chi[v \leq 0] ,$$

where $\nu(x,t) \geq 0$ is given by

$$\nu(x,t) = \begin{cases} \nu & , \ (x,t) \in [v < 0] \\ -w(x,t) & , \ (x,t) \in [v = 0] \end{cases} ,$$

and $\chi(\Sigma)$ denotes the characteristic function of the set Σ.

By virtue of (1.3), if $u \in W_2^{1,1}(\Omega_T)$ then also $v \in W_2^{1,1}(\Omega_T)$, and it will be enough to show the continuity of v in Ω_T.

Setting

$$\vec{a}(x,t,v,\nabla_x v) = \underset{\sim}{\vec{a}}(x,t,\beta_0^{-1}(v),\nabla_x\beta_0^{-1}(v))$$

$$b(x,t,v,\nabla_x v) = \underset{\sim}{b}(x,t,\beta_0^{-1}(v),\nabla_x\beta_0^{-1}(v)) \quad ,$$

identity (1.4) can be rewritten as

$$(1.5)\quad \int_\Omega (v(x,\tau)-\nu(x,\tau)\chi[v\le 0])\varphi(x,\tau)dx\,\Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega \{-(v(x,\tau)-\nu(x,\tau)\chi[v\le 0])\cdot$$

$$\cdot\frac{\partial\varphi}{\partial t} + \vec{a}(x,\tau,v,\nabla_x v)\cdot\nabla_x\varphi + b(x,\tau,v,\nabla_x v)\varphi\}dxd\tau = 0$$

$\forall\varphi\in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and all intervals $[t_0,t]\subset(0,T]$.

The above can be viewed as the weak formulation of

$$(1.6)\qquad \frac{\partial}{\partial t}\beta(v) - \operatorname{div}\vec{a}(x,t,v,\nabla_x v) + b(x,t,v,\nabla_x v) \ni 0 \quad \text{in } \mathcal{D}'(\Omega_T)$$

where $\beta(\cdot)$ is the maximal monotone graph

$$(1.7)\qquad \beta(r) = \begin{cases} r & r>0 \\ [0,-\nu] & r=0 \\ r-\nu & r<0\ . \end{cases}$$

In what follows we will assume $\beta(\cdot)$ is given as in (1.7).

Throughout the paper we will make the following assumptions on the coefficients $\vec{a} = (a_1, a_2, \ldots, a_N)$ and b.

$[A_1]$ $\qquad a_i,\ b\in C[\bar{\Omega}_T\times\mathbb{R}^{N+1}]\quad ,\quad i = 1, 2, \ldots, N\ .$

$[A_2]$ $$\sum_{i=1}^{N} a_i(x,t,v,\vec{p})p_i \ge C_0(|v|)|\vec{p}|^2 - \varphi_0(x,t)$$

$$|a_i(x,t,v,\vec{p})| \le \mu_0(|v|)|\vec{p}| + \varphi_1(x,t) \quad , \quad i = 1, 2, \ldots, N \ .$$

$$|b(x,t,v,\vec{p})| \le \mu_1(|v|)|\vec{p}|^2 + \varphi_2(x,t) \quad ,$$

where $C_0(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, decreasing, and strictly positive

$\mu_i(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing, $i = 0, 1,$

and the φ_i, $i = 0, 1, 2$ are non-negative and satisfy

$$\|\varphi_0, \varphi_2\|_{\hat{q},\hat{r},\Omega_T} \ , \ \|\varphi_1\|_{2\hat{q},2\hat{r},\Omega_T} \le \mu_2 \ .$$

Here μ_2 is a given constant and $\hat{q}$, $\hat{r}$ are positive numbers linked by

$$\frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \kappa_1$$

$$\hat{q} \in \left[\frac{N}{2(1-\kappa_1)}, \infty\right] \ , \quad \hat{r} \in \left[\frac{1}{1-\kappa_1}, \infty\right] \quad , \quad 0 < \kappa_1 < 1, \text{ for } N \ge 2$$

$$\hat{q} \in (1,\infty) \quad , \quad \hat{r} \in \left[\frac{1}{1-\kappa_1} \ , \ \frac{1}{1-2\kappa_1}\right] \ , \quad 0 < \kappa_1 < \frac{1}{2} \quad , \quad \text{for } N = 1$$

We can now state our main result.

Theorem 1: Let $[A_1] - [A_2]$ hold. Then every essentially bounded weak solution u of (1.1) is continuous in Ω_T.

If (1.1) is associated with an initial boundary value problem of Dirichlet or Neumann type then under suitable assumptions on the boundary conditions the continuity of u can be extended to the closure of Ω_T. For the precise statement of these results we refer to Section 5 Theorem 5.1, 5.2 and 5.3.

Remarks: (i) By the local nature of our arguments, in Theorem 1, the function u need not be defined in a cylindrical domain, since we can always reduce to this case by selecting in (1.4) test functions supported in cylindrical domains. Hence for the purpose of proving Theorem 1 one need only to assume that u is, locally essentially bounded in Q and that $u \in W^{1,1}_{2,loc}(Q)$.

(ii) It is of interest to know if Theorem 1 holds under the assumptions that u is essentially bounded in Ω_T and $u \in V^{1,0}_2(\Omega_T)$. A step in this direction can be found in Section 6.

(iii) Assumptions $[A_1]$ - $[A_2]$ are the same to those imposed in [18] to study the Hölder continuity of weak solutions of (1.1) with $\beta(r) = r$. In this connection in [22, 23] it is observed that the order of summability $\hat{p}$, $\hat{r}$ are optimal.

If $r \to \beta(r)$ is a monotone a.e. differentiable function satisfying (1.3), then the local Hölder continuity of the solution follows from the results of [18]. See also [8, 9] for the corresponding free-boundary problems.

We briefly comment on the regularity results at our knowledge available when $\beta(\cdot)$ is monotone and singular or degenerate.

For $N = 1$ and $b \equiv 0$, Fasano, Primicerio and Kamin showed in [15] that, under suitable assumptions on $\vec{a}(x,t,u,\nabla_x u)$, a generalized solution of (1.1) is locally Lipschitz-continuous in Ω_T. Hölder estimates where obtained by Cannon, Henry, Kotlov [10].

In [14] a similar result is obtained for a degenerate $\beta(\cdot)$ of the form

$$\beta(r) = \begin{cases} \beta_1(r) & r < 0 \\ 0 & r \geq 0 \end{cases} ,$$

where $\beta_1(\cdot)$ satisfies (1.3).

For $N > 1$, Caffarelli and Friedman [3] proved the continuity of nonnegative weak solutions of

$$\frac{\partial}{\partial t} u^{\alpha} - \Delta u = 0 \quad , \quad 0 < \alpha \le 1 \quad .$$

This result has been improved to the Hölder continuity by the same authors in [4].

Recently Caffarelli and Evans [2] have shown that weak solutions of

$$\frac{\partial}{\partial t} \beta(u) - \Delta u \ni 0 \quad \text{in} \quad \Omega_T$$

for $\beta(\cdot)$ given by

$$\beta(r) = \begin{cases} \beta_1 r & r > 0 \\ [-\nu, 0] & r = 0 \\ \beta_2 r & r < 0 \end{cases} ,$$

β_i, $i = 1, 2$ positive constants, are continuous. Their method of proof relies strongly on the properties of the Laplacian operator and the absence of lower order terms.

Our approach is completely different from the one in [2], and it is a natural continuation of ideas exposed in [12]. The method consists of a suitable modification of the parabolic version of the De Giorgi's estimates, as appearing in Ladijzenskaja-Solonnikov-Ural'tzeva [18].

The main idea of the proof can be described somehow euristically as follows. The function $(x,t) \to u(x,t)$ can be modified in a set of measure zero to yield a continuous representative out of the equivalence class $u \in W_2^{1,1}(\Omega_T)$ if for every $(x_0,t_0) \in \Omega_T$ there exists a family of nested and shrinking cylinders $Q_n(x_0,t_0)$ around (x_0,t_0), such that the essential oscillation ω_n of u in $Q_n(x_0,t_0)$, tends to zero as $n \to \infty$ in a way determined by the operator in (1.1) and the data.

The statement that a certain quantity, or function, depends upon the data will mean that it can be determined in terms of N, $C_0(\cdot)$, $\kappa_0(\cdot)$, $\omega_1(\cdot)$, φ_i, $i = 0, 1, 2$, $\hat{\delta}$, $\hat{r}$, κ_1, the jump ν of $\beta(\cdot)$ and the essential bound of u over Ω_T.

The paper is organized as follows. Section 2 contains some preliminary material and the derivation of a system of integral inequalities which will be the main tool in the proof of the theorem. Sections 3 and 4 are devoted to the proof of Theorem 1. The continuity up to the boundary is discussed in Section 5.

Finally in section 6 we show that if $u \in V_2^{1,0}(\Omega_T)$ is a weak solution of (1.1) which can be obtained as weak $V_2^{1,0}(\Omega_T)$-limit of certain approximations of (1.1) (in a sense to be made precise) then in fact the convergence takes place in the topology of the uniform convergence over compacts of Ω_T.

Since the arguments are technically heavy and the symbolism is quite complicated, an effort has been made to render the paper as self-contained as possible.

In view of this we have reproduced certain calculations already known from the literature.

I would like to thank M. Crandall for several helpful discussions on the subject.

2. Preliminary material and integral inequalities:

This section is devoted to the derivation of a system of integral inequalities which will be the main tool in the proof of Theorem 1.

Let $v \in L_{q,r}(\Omega_T)$ and $k \in \mathbb{R}$. Set

$$(v - k)^+ = \max\{(v - k);0\} \ ; \ (v - k)^- = \max\{-(v - k);0\} \quad .$$

It is obvious that $(v - k)^{\pm} \in L_{q,r}(\Omega_T)$ and it is known that if $v \in W_2^{1,1}(\Omega_T)$ so does $(v - k)^{\pm}$, (see [19]).

With $B(R)$ we denote a ball of radius R in $\mathbb{R}^N$ and if $x \to v(x)$ is defined in Ω, and $B(R) \subset \Omega$ we set

$$A^+_{k,R} \equiv \{x \in B(R) \,|\, v(x) > k\}$$

$$A^-_{k,R} \equiv \{x \in B(R) \,|\, v(x) < k\} \quad .$$

Also let κ_N denote the measure of the surface of the unit sphere so that meas $B(R) = \kappa_N R^N$.

From now on $(x,t) \to v(x,t)$ will denote a weak solution of (1.6), and M is a positive real number such that

$$\operatorname*{ess\,sup}_{\Omega_T} |v| \leq M \quad .$$

We will think of $(x,t) \to v(x,t)$ as an arbitrarily selected and fixed representative out of the equivalence class v, so that the map $(x,t) \to v(x,t) \in \overline{\mathbb{R}}$ is well defined $\forall (x,t) \in \Omega_T$.

We will derive a system of inequalities for v by making particular selections of the test function φ in the identity (1.5).

First we observe that since $v \in W_2^{1,1}(\Omega_T)$, (1.5) can be rewritten as

$$(2.1)\quad -\int_\Omega \nu(x,\tau)\chi[v\le 0]\varphi\, dx\,\Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega \nu(x,\tau)\chi[v\le 0]\frac{\partial}{\partial t}\varphi\, dxd\tau +$$

$$+\int_{t_0}^{t}\int_\Omega \{\frac{\partial}{\partial t} v\varphi + \vec{a}(x,\tau,v,\nabla_x v)\cdot\nabla_x\varphi + b(x,\tau,v,\nabla_x v)\varphi\}dxd\tau = 0$$

$\forall\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$, and any interval $[t_0,t] \subset (0,T]$.

Next we construct the test functions in (2.1).

Let $\sigma_1, \sigma_2 \in (0,1)$ and consider the concentric balls $B(R)$ and $B(R - \sigma_1 R)$, and the cylinders $Q(R,\lambda) \equiv B(R) \times [t_0,t_0 + \lambda]$ and $Q(R - \sigma_1 R,\lambda - \sigma_2\lambda) \equiv B(R - \sigma_1 R) \times [t_0 + \sigma_2\lambda, t_0 + \lambda]$, $\lambda > 0$.

Define cutoff functions in $Q(R,\lambda)$ as follows

(a) $\zeta \in C^\infty[Q(R,\lambda)]$ such that $\zeta(x,t)|_{\partial B(R)} = 0$ $\forall t \in [t_0,t_0 + \lambda]$, $\zeta(x,t_0) = 0$ $\forall x \in B(R)$ and $\zeta(x,t) = 1$, $(x,t) \in Q(R - \sigma_1 R,\lambda - \sigma_2\lambda)$, $\frac{\partial}{\partial t}\zeta \ge 0$, $|\nabla_x\zeta| \le (\sigma_1 R)^{-1}$; $|\frac{\partial}{\partial t}\zeta| \le (\sigma_2\lambda)^{-1}$.

(b) $\zeta \in C_0^\infty(B(R))$ such that $\zeta(x) = 1$, $x \in B(R - \sigma_1 R)$, $|\nabla\zeta| \le (\sigma_1 R)^{-1}$.

For any cylinder $Q(R,\lambda) \subset \Omega_T$ we make the following selections of test function in (2.1)

$$\varphi = \pm(v - k)^{\pm}\zeta^2$$

where $k \in \mathbb{R}$ satisfies

$$(2.2)\qquad \operatorname*{ess\,sup}_{Q(R,\lambda)} (v - k)^{\pm} \le \delta$$

for some $\delta > 0$ to be selected, and $(x,t) \to \zeta(x,t)$ is either as in (a) or as in (b).

For simplicity of notation we set

$$- \int_\Omega \nu(x,\tau)\chi[v\le 0][\pm(v-k)^{\pm}]\zeta^2 dx\Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega \nu(x,\tau)\chi[v\le 0]\,\frac{\partial}{\partial t}\,[\pm(v-k)^{\pm}\zeta^2]dxd\tau$$

$$= -\,\Phi^{\pm}(k,t_0,t,\zeta) \qquad , \quad t \in (t_0,t_0+\lambda)$$

and transform and estimate the remaining parts of (2.1) as follows

$$I = \int_{t_0}^{t}\int_\Omega \pm\frac{\partial}{\partial t}\, v(v-k)^{\pm}\,\zeta^2\,dxd\tau = \frac{1}{2}\int_{t_0}^{t}\int_\Omega \frac{\partial}{\partial t}\,[(v-k)^{\pm}]^2\,\zeta^2\,dxd\tau =$$

$$= \frac{1}{2}\,\|(v-k)^{\pm}\,\zeta\|^2_{2,\Omega}(\tau)\Big|_{t_0}^{t} - \int_{t_0}^{t}\int_\Omega [(v-k)^{\pm}]^2\,\zeta\,\frac{\partial}{\partial t}\,\zeta\,dxd\tau \quad .$$

To estimate the last two terms in (2.1) we take in account the assumptions $[A_1]$ - $[A_2]$. We have

$$J_1 = \int_{t_0}^{t}\int_\Omega \vec{a}(x,\tau,v,\nabla_x v)\cdot\nabla_x[\pm(v-k)^{\pm}\,\zeta^2]dxd\tau =$$

$$= \sum_{i=1}^{N}\int_{t_0}^{t}\int_\Omega a_i(x,\tau,v,\nabla_x v)\,\frac{\partial}{\partial x_i}\,[\pm(v-k)^{\pm}]\zeta^2\,dxd\tau +$$

$$+ \int_{t_0}^{t}\int_\Omega \vec{a}(x,\tau,v,\nabla_x v)[\pm(v-k)^{\pm}]\cdot\nabla_x\zeta^2\,dxd\tau \ge$$

$$\ge \int_{t_0}^{t}\int_\Omega C_0(|v|)|\nabla_x(v-k)^{\pm}|^2\,\zeta^2\,dxd\tau - \int_{t_0}^{t}\int_\Omega \varphi_0\zeta^2\chi[(v-k)^{\pm}>0]dxd\tau$$

$$- 2\int_{t_0}^{t}\int_\Omega \mu_0(|v|)|\nabla_x(v-k)^{\pm}|(v-k)^{\pm}\,\zeta|\nabla_x\zeta|dxd\tau\, -$$

$$- 2\int_{t_0}^{t}\int_\Omega \varphi_1\,(v-k)^{\pm}\,\zeta|\nabla_x\zeta|dxd\tau \quad .$$

$$J_2 = \int_{t_0}^{t} \int_{\Omega} b(x,\tau,v,\nabla_x v)[\pm(v-k)^{\pm}]\zeta^2\, dxd\tau \geq -\int_{t_0}^{t} \int_{\Omega} \mu_1(|v|)|\nabla(v-k)^{\pm}|\zeta^2 \cdot$$

$$(v-k)^{\pm}\ \zeta^2\, dxd\tau - \int_{t_0}^{t} \int_{\Omega} \varphi_2\ (v-k)^{\pm}\ \zeta^2\, dxd\tau$$

Since $\underset{\Omega_T}{\text{ess sup}}\ |v| \leq M$, from the assumptions on $C_0(\cdot)$ and $\mu_i(\cdot)$ we see that $C_0(|v|) \geq C_0(M)$, $\mu_i(|v|) \leq \mu_i(M)$, $i = 0, 1$. From the Cauchy inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ we have

$$2\int_{t_0}^{t} \int_{\Omega} \mu_0(M)|\nabla_x(v-k)^{\pm}|\ (v-k)^{\pm}\ \zeta|\nabla_x\zeta|dxd\tau \leq$$

$$\leq \varepsilon \int_{t_0}^{t} \int_{\Omega} |\nabla_x(v-k)^{\pm}|^2\ \zeta^2\ dxd\tau + \varepsilon^{-1}\mu_0^2(M) \int_{t_0}^{t} \int_{\Omega} [(v-k)^{\pm}]^2|\nabla_x\zeta|^2 dxd\tau$$

and

$$2\int_{t_0}^{t} \int_{\Omega} \varphi_1\ (v-k)^{\pm}\zeta|\nabla_x\zeta|\, dxd\tau \leq \int_{t_0}^{t} \int_{\Omega} [(v-k)^{\pm}]^2|\nabla_x\zeta|^2 dxd\tau +$$

$$+\int_{t_0}^{t} \int_{\Omega} \varphi_1^2\ \zeta^2\chi[(v-k)^{\pm} > 0]\, dxd\tau$$

In estimating the integrals in J_2 we recall (2.2), so that combining the estimates for J_1 and J_2 we obtain

$$J_1 + J_2 \geq (c_0(M) - \varepsilon - \delta\mu_1(M)) \int_{t_0}^{t} \int_{\Omega} |\nabla_x (v - k)^{\pm}|^2 \zeta^2 \, dx d\tau -$$

$$- (\varepsilon^{-1}\mu_0^2(M) + 1) \int_{t_0}^{t} \int_{\Omega} [(v - k)^{\pm}]^2 |\nabla_x \zeta|^2 dx d\tau -$$

$$- \int_{t_0}^{t} \int_{\Omega} [\varphi_0 + \delta\varphi_2 + \varphi_1^2]\zeta^2 \chi[(v - k)^{\pm} \neq 0] dx d\tau$$

Next we estimate the last integral above in terms of the measure of the set $[(v - k)^{\pm} > 0]$, by employing the assumption $[A_2]$. We set

$$A^{\pm}_{k,R}(\tau) \equiv \{x \in B(R) \,|\, (v - k)^{\pm}(x,\tau) > 0\} \quad .$$

Then by the Hölder inequality

$$J^* = \int_{t_0}^{t} \int_{\Omega} [\varphi_0 + \delta\varphi_2 + \varphi_1^2]\zeta^2 \chi[(v - k)^{\pm} > 0] dx d\tau \leq$$

$$\leq \max[1,\delta] \, \|\varphi_0 + \varphi_2 + \varphi_1^2\|_{\hat{q},\hat{r},\Omega_T} \left\{ \int_{t_0}^{t} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{\hat{q}-1}{\hat{q}} \frac{\hat{r}}{\hat{r}-1}} d\tau \right\}^{\frac{\hat{r}-1}{\hat{r}}} .$$

Setting

$$(2.3) \qquad q = \frac{2\hat{q}(1 + \kappa)}{\hat{q} - 1} \quad , \quad r = \frac{2\hat{r}(1 + \kappa)}{\hat{r} - 1} \quad , \quad \kappa = \frac{2\kappa_1}{N} \quad ,$$

$$J^* \leq \max[1,\delta] \, \|\varphi_0 + \varphi_2 + \varphi_1^2\|_{\hat{q},\hat{r},\Omega_T} \left\{ \int_{t_0}^{t} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)} .$$

Since $\frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \kappa_1$ we have

(2.4) $$\frac{1}{r} + \frac{N}{2q} = \frac{N}{4} ,$$

and the admissible range of r and q is

(2.5) $$\begin{cases} q \in (2, \frac{2N}{N-2}] , & r \in [2,\infty) \quad \text{for} \quad N \geq 3 \\ q \in (2,\infty) , & r \in (2,\infty) \quad \text{for} \quad N = 2 \\ q \in (2,\infty) , & r \in [4,\infty) , \quad \text{for} \quad N = 1 . \end{cases}$$

In the estimate of $J_1 + J_2$ we choose

(2.6) $$\varepsilon = \frac{C_0(M)}{4} \qquad \delta = \min \{ \frac{C_0(M)}{4\mu_1(M)} , 1\}$$

so that collecting all the previous estimates we obtain the inequalities

$$\|(v-k)^{\pm}\zeta\|^2_{2,\Omega}(t) + \int_{t_0}^{t} \int_{\Omega} |\nabla_x (v-k)^{\pm}|^2 \zeta^2 \, dxd\tau \leq$$

$$\leq \|(v-k)^{+}\zeta\|^2_{2,\Omega}(t_0) + \gamma \int_{t_0}^{t_0+\lambda} \int_{\Omega} [(v-k)^{\pm}]^2 (|\nabla_x \zeta|^2 + \zeta \, |\frac{\partial}{\partial t} \zeta|) dxd\tau +$$

$$+ \gamma \left\{ \int_{t_0}^{t_0+\lambda} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)} + \Phi^{\pm}(k,t_0,t,\zeta) , \ \forall t \in [t_0,t_0+\lambda] ,$$

where γ is a constant depending only upon the data.

These inequalities are valid for every cylinder $Q(R,\lambda) \subset \Omega_T$ and every $k \in \mathbb{R}$, satisfying (2.2) with the choice (2.6) of the parameter δ.

If we select the cutoff function $(x,t) \to \zeta(x,t)$ as in (a) we see that there exists a constant γ, dependent only upon the data, such that

$$(2.7) \qquad \left|(v-k)^{\pm}\right|^2_{V_2^{1,0}[B(R-\sigma_1 R)\times(t_0+\sigma_2\lambda, t_0+\lambda)]} \le$$

$$\le \gamma[(\sigma_1 R)^{-2} + (\sigma_2\lambda)^{-1}] \, \|(v-k)^{\pm}\|^2_{2,Q(R,\lambda)} +$$

$$+ \gamma \left\{ \int_{t_0}^{t_0+\lambda} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)} +$$

$$+ \sup_{t\in[t_0, t_0+\lambda]} \Phi_a^{\pm}(k, t_0, t) \; .$$

where $\Phi_a^{\pm}(k,t_0,t)$ coincides with $\Phi^{\pm}(k,t_0,t,\zeta)$ if $\zeta(x,t)$ is selected as in (a).

Choosing now $x \to \zeta(x)$ as in (b) we have

$$(2.8) \qquad \|(v-k)^{\pm}\|^2_{2,A^{\pm}_{k,R-\sigma_1 R}(t)} + \|\nabla_x(v-k)^{\pm}\|^2_{2,Q(R-\sigma_1 R,\lambda)} \le$$

$$\le \|(v-k)^{\pm}\|^2_{2,A^{\pm}_{k,R}(t_0)} + \gamma(\sigma_1 R)^{-2}\|(v-k)^{\pm}\|^2_{2,Q(R,\lambda)} +$$

$$+ \gamma \left\{ \int_{t_0}^{t_0+\lambda} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)} + \Phi_b^{\pm}(k,t_0,t), \; t \in [t_0, t_0+\lambda]$$

with the obvious definition of $\Phi_b^{\pm}(k,t_0,t)$.

Roughly speaking, inequalities (2.7) - (2.8) supply some local control on that part of the graph of v which lies above (below) the hyperplane $v = k$.

Consider a region $\mathcal{O} \subset \Omega_T$ such that

$$\operatorname{meas}[(x,t) \in \mathcal{O} \mid v(x,t) \le 0] = 0 \quad .$$

Then for every cylinder $Q(\rho,\lambda) \subset \mathcal{O}$, $\Phi^{\pm}(k,t_0,t) \equiv 0$, hence by choosing the cutoff function $\zeta(x,t)$ as in (a) and as in (b), we see that the functions $(x,t) \to (v - k)^{\pm}(x,t)$ satisfy inequalities (7.1) - (7.2) of [18] page 110. By virtue of the embedding theorem 7.1 of [18] page 120, this implies that $(x,t) \to v(x,t)$ is Hölder continuous in every region $\mathcal{O}' \subset \mathcal{O}$. An analogous argument holds for regions $\mathcal{O}$ such that $\operatorname{meas}[(x,t) \in \mathcal{O} \mid v(x,t) \ge 0] = 0$.

Because of the presence of the term $\Phi^{\pm}(k,t_0,t)$, we do not expect that inequalities (2.7) - (2.8) imply the continuity of the solution, without additional informations contained in the identity (2.1). This is the role of the next two lemmas.

Let $\theta \in \mathbb{R}^+$ and consider the cylinders $Q(R,\theta R^2) \equiv B(R) \times [t_0,t_0 + \theta R^2]$ and $Q(R - \sigma_1 R,\theta R^2) \equiv B(R - \sigma_1 R) \times [t_0,t_0 + \theta R^2]$.

<u>Lemma 2.1</u>. Let $\zeta(x)$ be a cutoff function in $Q(R,\theta R^2)$ chosen as in (b). Then there exists a constant $C(M,\theta,\nu)$ such that

$$\iint_{Q(R,\theta R^2)} |\nabla_x v|^2 \, \zeta^2(x)\,dx\,dt \le \frac{C(M,\theta,\nu)}{\sigma_1^2} \kappa_N R^N \quad .$$

<u>Proof</u>: In (2.1) select the test function $\varphi = e^{\lambda v}\zeta^2(x)$, where $\lambda > 0$ will be chosen later. For all $t \in [t_0,t_0 + \theta R^2]$ we have

$$- \int_{\Omega} \nu(x,\tau)\chi[v \le 0]\varphi \, dx \Big|_{t_0}^{t} - \int_{t_0}^{t} \int_{\Omega} \nu(x,\tau)\chi[v \le 0] \frac{\partial}{\partial t} v^- e^{-\lambda v^-} \zeta^2(x)\,dx\,d\tau$$

$$= - \int_{\Omega} \nu(x,\tau)\chi[v \le 0]e^{-\lambda v^-} \zeta^2(x)\,dx \Big|_{t_0}^{t} + \nu \int_{t_0}^{t} \int_{\Omega} \frac{\partial}{\partial t} (e^{-\lambda v^-} - 1) \cdot \zeta^2(x)\,dx\,d\tau$$

This term, and the term $\left|\int_{t_0}^{t}\int_{\Omega}\frac{\partial}{\partial t} v\,\varphi\,dxd\tau\right|$, can be easily dominated in terms of $\bar{C}\kappa_N R^N$ where $\bar{C}$ depends upon M, λ and ν.

On the other hand using the assumptions $[A_1]$ - $[A_2]$, standard calculations yield

$$\int_{t_0}^{t}\int_{\Omega}\{\vec{a}(x,\tau,v,\nabla_x v)\nabla_x\varphi + b(x,\tau,v,\nabla_x v)\varphi\}dxd\tau \geq$$

$$\geq [\lambda C_0(M) - \varepsilon - \mu_1(M)]\int_{t_0}^{t}\int_{\Omega} e^{\lambda v}|\nabla_x v|^2\,\zeta^2(x)dxd\tau -$$

$$- \quad e^{\lambda M}\int_{t_0}^{t}\int_{\Omega}(\lambda\varphi_0 + \varphi_2)\zeta^2(x)dxd\tau - 2e^{\lambda M}\int_{t_0}^{t}\int_{\Omega}\varphi_1\zeta|\nabla\zeta|dxd\tau -$$

$$- \frac{4}{\varepsilon}\mu_0^2(M)e^{\lambda M}\int_{t_0}^{t}\int_{\Omega}|\nabla\zeta|^2\,dxd\tau \quad .$$

Selecting $\varepsilon = \mu_1(M)$ and $\lambda = \dfrac{4}{C_0(M)}\mu_1(M)$ we conclude that there exists a constant $\bar{\bar{C}}$ depending upon M such that

$$e^{-\lambda M}\iint_{Q(R,\theta R^2)}|\nabla_x v|^2\,\zeta^2(x)dxd\tau \leq \bar{C}\kappa_N R^N +$$

$$+ \bar{\bar{C}}\left\{\int_{t_0}^{t_0+\theta R^2}\int_{\Omega}[\varphi_0 + \varphi_2 + \varphi_1^2]\zeta^2(x)dxd\tau + \int_{t_0}^{t_0+\theta R^2}\int_{\Omega}|\nabla\zeta|^2\,dxd\tau\right\} \quad .$$

We recall that $|\nabla\zeta| < (\sigma_1 R)^{-1}$ and treat the integral involving the φ_i $i = 0, 1, 2$ as previously, to obtain

$$\iint_{Q(R,\theta R^2)}|\nabla_x v|^2\,\zeta^2(x)dxd\tau \leq \frac{C(M,\lambda,\nu)}{\sigma_1^2}\kappa_N R^N \quad .$$

This inequality will be employed to prove the following lemma.

Lemma 2.2: Let $k \in \mathbb{R}^+, \mu \geq \operatorname*{ess\,sup}_{Q(R,\theta R^2)} (v-k)^+$ and $0 < \eta < \mu$

Set

$$\psi(x,t) = \ell n^+ \left[\frac{\mu}{\mu - (v-k)^+ + \eta} \right] = \max \left\{ \ell n \left[\frac{\mu}{\mu - (v-k)^+ + \eta} \right] ; 0 \right\} ,$$

then there exists a constant $C = C(\theta)$ such that for all $t \in [t_0, t_0 + \theta R^2]$

$$\int_{B(R-\sigma_1 R)} \psi^2(x,t)dx \leq \int_{B(R)} \psi^2(x,t_0)dx +$$

$$+ \frac{C}{\sigma_1^2} (1 + \ell n \frac{\mu}{\eta})(1 + \frac{R^{N\kappa}}{\eta^2})\kappa_N R^N \quad .$$

Remark: *For simplicity of notation we will use the same symbol* ψ for $\psi(x,t)$ and $\tilde{\psi}(v(x,t))$. In what follows ψ' will mean $\frac{\partial}{\partial v}\tilde{\psi}$.

Proof: In (2.1) we select $\varphi = (\psi^2)' \zeta^2(x)$, where $\zeta(x)$ is chosen as in (b). It is apparent that $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$, and that $(\psi^2)'' = 2(1+\psi)(\psi')^2$. Since $(\psi^2)'$ vanishes at those points $(x,t) \in \Omega_T$ where $(v-k)^+ \leq \eta$, and $\eta > 0$, the terms involving $\nu(x,t)\chi[v \leq 0]$ in (2.1) does not give any contribution. The term involving $\frac{\partial}{\partial t} v$ gives

$$\int_{t_0}^{t} \int_{\Omega} \frac{\partial}{\partial t} v(\psi^2)' \zeta^2(x)dxd\tau = \int_{\Omega} \psi^2(x,\tau)\zeta^2(x)dx \Big|_{t_0}^{t} \quad .$$

We estimate the remaining terms as follows

$$\int_{t_0}^{t}\int_{\Omega} \tilde{a}(x,\tau,v,\nabla_x v)\{2(1+\psi)(\psi')^2\, \nabla_x v\, \zeta^2(x) + (\psi^2)'\, \nabla\zeta^2(x)\}dxd\tau \geq$$

$$\geq 2C_0(M)\int_{t_0}^{t}\int_{\Omega} (1+\psi)|\nabla_x\psi|^2\, \zeta^2(x)dxd\tau -$$

$$-2\int_{t_0}^{t}\int_{\Omega} \varphi_0(x,\tau)(1+\psi)(\psi')^2\, \zeta^2(x)dxd\tau -$$

$$-4\mu_0(M)\int_{t_0}^{t}\int_{\Omega} \psi^{\frac{1}{2}}|\nabla_x\psi|\zeta(x)\psi^{\frac{1}{2}}|\nabla\zeta|dxd\tau -$$

$$-\int_{t_0}^{t}\int_{\Omega} \varphi_1(x,\tau)(\psi^2)'|\nabla\, \zeta^2(x)|dxd\tau \geq$$

$$\geq (2C_0(M)-\varepsilon)\int_{t_0}^{t}\int_{\Omega} (1+\psi)|\nabla_x\psi|^2\, \zeta^2(x)dxd\tau -$$

$$-\int_{t_0}^{t}\int_{\Omega} \{(\psi')^2\varphi_0(x,\tau)(1+\psi)\zeta^2(x) + \varphi_1(x,\tau)|\nabla\zeta^2|(\psi^2)'\}dxd\tau -$$

$$-\int_{t_0}^{t}\int_{\Omega} \varepsilon^{-1}(4\mu_0)^2(M)\psi|\nabla\zeta|^2\, dxd\tau \quad .$$

For the lower order terms we have

$$\int_{t_0}^{t}\int_{\Omega} |b(x,\tau,v,\nabla_x v)(\psi^2)'\, \zeta^2(x)|dxd\tau \leq 2\mu_1(M)\int_{t_0}^{t}\int_{\Omega} \psi\psi'|\nabla_x v|^2\, \zeta^2(x)dxd\tau$$

$$+\int_{t_0}^{t}\int_{\Omega} \varphi_2(x,\tau)(\psi^2)'\, \zeta^2(x)dxd\tau \quad .$$

Since $\psi\psi'|\nabla_x v|^2 = \psi^{\frac{1}{2}}|\nabla_x\psi|\psi^{\frac{1}{2}}|\nabla_x v|$ we have

$$2\mu_1(M)\int_{t_0}^{t}\int_{\Omega}\psi\psi'|\nabla_x v|^2\,\zeta^2(x)\,dxd\tau \le$$

$$\le \varepsilon\int_{t_0}^{t}\int_{\Omega}(1+\psi)|\nabla_x\psi|^2\,\zeta^2(x)\,dxd\tau +$$

$$+\ \varepsilon^{-1}4\mu_1^2(M)\int_{t_0}^{t}\int_{\Omega}\psi|\nabla_x v|^2\,\zeta^2(x)\,dxd\tau \quad .$$

Collecting the previous estimates gives

$$\int_{\Omega}\psi^2\,\zeta^2(x)\,dx\,\Big|_{t_0}^{t} + [2C_0(M) - 2\varepsilon]\int_{t_0}^{t}\int_{\Omega}(1+\psi)|\nabla_x\psi|^2\,\zeta^2(x)\,dxd\tau \le$$

$$\le 2\int_{t_0}^{t}\int_{\Omega}\{(\psi')^2(1+\psi)[\varphi_0+\varphi_1^2] + \psi\psi'\,\varphi_2\}\zeta^2(x)\,dxd\tau +$$

$$+\ [2+\varepsilon^{-1}\,4(4\mu_0^2(M)+\mu_1^2(M))]\int_{t_0}^{t}\int_{\Omega}\psi\{|\nabla_x v|^2\,\zeta^2(x) + |\nabla_x\zeta|^2\}dxd\tau \quad .$$

We select $\varepsilon = C_0(M)$ and observe that since $\eta < \mu$, $\psi' < \frac{1}{\eta}$ and $\psi \le \ell n\,\frac{\mu}{\eta}$. Moreover we recall that $|\nabla\zeta| < (\sigma_1 R)^{-1}$ and lemma 2.1 in estimating the last term in the inequality above. This yields the existence of a constant $\tilde{C}(M,\nu,\theta)$ such that

$$\int_{\Omega} \psi^2 \zeta^2(x)\,dx \Big|_{t_0}^{t} \le \frac{\bar{C}}{\varepsilon_1^2}\left(1 + \ln\frac{\mu}{\eta}\right) \left\{ \frac{1}{\eta^2} \int_{t_0}^{t_0+\theta R^2} \int_{B(R)} [\varphi_0 + \varphi_2 + \varphi_1^2]\,dx\,d\tau \right.$$

$$\left. + \kappa_N R^N \right\} \le$$

$$\le \frac{\bar{\bar{C}}}{\varepsilon_1^2}\left(1 + \ln\frac{\mu}{\eta}\right) \left\{ \frac{1}{\eta^2} \|\varphi_0 + \varphi_2 + \varphi_1^2\|_{\hat{q},\hat{r},\Omega_T}\, \theta^{\frac{2}{r}(1+\kappa)}\, \kappa_N^{\frac{2}{q}(1+\kappa)-1} \,\cdot \right.$$

$$\left. \cdot\, \kappa_N R^{N+N\kappa} + \kappa_N R^N \right\} .$$

This proves the lemma.

<u>Remarks</u>: (i) If $k \le 0$ and $\bar{\mu} \ge \operatorname{ess\,sup}_{Q_R^\theta} (v - k)^-$, then an analogous lemma holds for

$$\bar{\psi}(x,t) = \ln^+ \left[\frac{\bar{\mu}}{\bar{\mu} - (v - k)^- + \eta} \right] \quad ; \; 0 < \eta < \bar{\mu} .$$

The proof is the same except for minor changes.

(ii) The proof shows that $C(\theta)$ increases with θ. We will use lemma 2.2 with $0 < \theta \le 1$ and $C(\theta)$ replaced by $C = C(1)$.

We report a lemma due to De Giorgi [11] which will be used as we proceed.

<u>Lemma 2.3 (De Giorgi)</u>: Let $v \in W_1^1(B(R))$ and let k, ℓ be real numbers such that $\ell > k$. Then

$$(2.9) \qquad (\ell - k)\,\mathrm{meas}\, A_{\ell,R} \le D \frac{R^{N+1}}{\mathrm{meas}(B(R) \setminus A_{k,R})} \int_{A_{k,R} \setminus A_{\ell,R}} |\nabla v|\,dx ,$$

where D is a constant depending only upon the dimension N.

Inequality (2.9) holds for domains other than balls. We refer to [18,19] for details noticing for later use that it is valid for convex domains.

Finally if Q is a cylindrical domain in $\mathbb{R}^{N+1}$, $\overset{\circ}{V}{}_2^{1,0}(Q)$ denotes the subspace of $V_2^{1,0}(Q)$ of functions whose trace is zero on the lateral boundary of Q, equipped with the same norm as $V_2^{1,0}(Q)$.

The proof of the following embedding lemma can be found in [18] page 74-77.

<u>Lemma 2.4</u>: If $v \in \overset{\circ}{V}{}_2^{1,0}(Q)$ then $v \in L_{q,r}(Q)$ where q,r satisfy (2.4) - (2.5). Moreover there exists a constant β depending only upon the dimension N such that

$$\|v\|_{q,r,Q} \leq \beta |v|_{\overset{\circ}{V}{}_2^{1,0}(Q)} . \tag{2.10}$$

If $q = r = 2$ then

$$\|v\|_{2,Q} \leq \beta[\text{meas}[|v| \neq 0] \cap Q]^{\frac{1}{N+2}} |v|_{\overset{\circ}{V}{}_2^{1,0}(Q)} . \tag{2.11}$$

If $v \in V_2^{1,0}(Q)$ then (2.10) is still valid. Moreover if $p = r = 2$ and if $Q \equiv \Omega \times (0,T)$

$$\|v\|_{2,Q} \leq C[\text{meas}[|v| \neq 0] \cap Q]^{\frac{1}{N+2}} |v|_{V_2^{1,0}(Q)} \tag{2.12}$$

where $C = 2\beta + (T^{\frac{N}{2}} \text{meas}\, \Omega^{-1})^{\frac{1}{N+2}}$.

3. The main proposition:

Throughout this section we let $(x_0,t_0) \in \Omega_T$ $t_0 > 0$ and for $R > 0$, Q_R will denote the cylinder

$$Q_R \equiv \{|x - x_0| < R\} \times [t_0 - R^2, t_0] \quad .$$

Let $R_0 < \frac{1}{2}$ be so small that $Q_{2R_0} \subset \Omega_T$, set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_{2R_0}} v \quad ; \quad \mu^- = \operatorname*{ess\,inf}_{Q_{2R_0}} v \quad ,$$

and denote with ω any positive real number such that

$$2M \geq \omega \geq \operatorname*{ess\,osc}_{Q_{2R_0}} v = \mu^+ - \mu^- \quad .$$

For $k \in \mathbb{R}$ and $0 < R \leq 2R_0$ we set

$$Q_R^+(k) \equiv \{(x,t) \in Q_R | v(x,t) > k\}$$

$$Q_R^-(k) \equiv \{(x,t) \in Q_R | v(x,t) < k\} \quad .$$

Finally we let s *denote the smallest positive integer such that*

(3.1) $$\frac{2M}{2^s} < \delta \quad , \quad s \geq 2 \quad ,$$

where δ is the number introduced in (2.6).

The goal of this section is to prove the following result

Proposition 3.1: Let ω be any positive number such that

$$2M \geq \omega \geq \operatorname*{ess\,osc}_{Q_{2R_0}} v \quad .$$

Then there exist numbers $s_0 \in \mathbb{N}$, A, $a > 1$, $h > 1$, $\xi_* < 1$ such that

$$\operatorname*{ess\,osc}_{Q_{R_*}} v \leq \omega\left(1 - \frac{1}{2^{s_0 + A/\omega^a}}\right)$$

where $R_* = \xi_*(2R_0)^h$, provided that

$$\frac{\omega}{2^{s_0 + A/\omega^a}} \geq (2R_0)^{\frac{N\kappa}{2}} .$$

The numbers s_0, A, a, h, ξ_* depend uniquely upon the data and not upon R_0 nor ω.

Without loss of generality we may assume that

$$|\mu^-| \leq \mu^+ . \tag{3.2}$$

If the reverse inequality holds the arguments are similar. Also we will assume that

$$\operatorname*{ess\,osc}_{Q_{2R_0}} v = \mu^+ - \mu^- > \frac{\omega}{2^{s-1}} , \tag{3.3}$$

and treat later the case $\mu^+ - \mu^- \leq \frac{\omega}{2^{s-1}}$.

Notice that (3.2) - (3.3) imply that

$$\mu^+ - \frac{\omega}{2^s} > \left|\frac{\omega}{2^s} + \mu^-\right| \geq 0 . \tag{3.4}$$

Observe moreover that we may assume

$$H \equiv \operatorname*{ess\,sup}_{Q_{R_0}} \left(v - \left(\mu^- + \frac{\omega}{2^s}\right)\right)^- > \frac{\omega}{2^{s+1}} . \tag{Γ}$$

Indeed if (*) is violated

$$- \operatorname*{ess\,inf}_{Q_{R_0}} v \leq - \mu^- - \frac{\omega}{2^s} + \frac{\omega}{2^{s+1}}$$

and adding $\operatorname*{ess\,sup}_{Q_{R_0}} v$ on the left hand side and μ^+ on the right hand side we obtain

$$\operatorname*{ess\,osc}_{Q_{R_0}} v \leq \omega(1 - \frac{1}{2^{s+1}})$$

and Proposition 3.1 becomes trivial.

Proposition 3.1 will be a consequence of a series of lemmas which we state and prove independently.

Lemma 3.1: There exists a number c_0 depending only upon the data and independent of ω and R_0 such that if

$$\text{meas } Q^-_{R_0}(\mu^- + \frac{\omega}{2^s}) \leq c_0 \, \omega^{\frac{2\kappa_1}{N+2\kappa_1}} \kappa_N R_0^{N+2} ,$$

then either

(i) $$H = \operatorname*{ess\,sup}_{Q_{R_0}} (v - (\mu^- + \frac{\omega}{2^s}))^- \leq R_0^{\frac{N\kappa}{2}}$$

or

(ii) $$\text{meas } Q^-_{\frac{R_0}{2}} (\mu^- + \frac{\omega}{2^s} - \frac{1}{2} H) = 0 .$$

Here κ_1 is the number appearing in the assumptions $[A_1] - [A_2]$.

Proof of lemma 3.1: Consider inequalities (2.7) for the function $(x,t) \to (v-k)^-$, $\mu^- \le k \le \mu^- + \frac{\omega}{2^s}$ in the cylinders Q_R, $0 < R \le R_0$. Notice that in view of (3.1)

$$\operatorname*{ess\,sup}_{Q_R} (v-k)^- \le \frac{\omega}{2^s} < \delta$$

so that the use of (2.7) for $\mu^- \le k \le \mu^- + \frac{\omega}{2^s}$ is justified.

We estimate $\Phi_a^-(k,t_0 - R^2,t_0)$ in (2.7) by distinguishing the cases of $k \le 0$ and $k > 0$.

If $k \le 0$ then $\Phi_a^-(k,t_0 - R^2,t_0) = 0$.

If $k > 0$ we have

$$\Phi_a^-(k,t_0 - R^2,t_0) = -\int_\Omega \nu(x,\tau)\chi[v \le 0]v^- \zeta^2(x,\tau)dx \Big|_{t_0-R^2}^{t_0} -$$

$$- k\int_\Omega \nu(x,\tau)\chi[v \le 0]\zeta^2(x,\tau)dx \Big|_{t_0-R^2}^{t_0} + \int_{t_0-R^2}^{t_0}\int_\Omega \nu(x,\tau)\chi[v \le 0](v-k)^-$$

$$\frac{\partial}{\partial t}\zeta^2(x,\tau)dxd\tau + \int_{t_0-R^2}^{t_0}\int_\Omega \nu\frac{\partial}{\partial t}v^- \zeta^2(x,\tau)dxd\tau \le$$

$$\le 2\nu\int_{t_0-R^2}^{t_0}\int_\Omega (v-k)^-\frac{\partial}{\partial t}\zeta(x,\tau)dxd\tau - \nu\int_{t_0-R^2}^{t_0}\int_\Omega v^-(x,\tau)\frac{\partial}{\partial t}\zeta^2(x,\tau)dxd\tau \quad .$$

If $(x,t) \to \zeta(x,t)$ is selected as in (a), $\zeta_t \ge 0$ so that $v^- \zeta_t \ge 0$ and

$$\Phi_a^-(k,t_0 - R^2,t_0) \le \frac{2\nu}{\sigma_2 R^2}\int\int_{Q_R} (v-k)^- dxd\tau \quad .$$

Inequalities (2.7) now read

$$\text{(3.5)} \qquad |(v-k)^-|^2_{V_2^{1,0}[B(R-\sigma_1 R)\times(t_0-(1-\sigma_2)R^2,t_0)]} \le$$

$$\le \gamma[(\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}]\, \|(v-k)^-\|^2_{2,Q_R} +$$

$$+ \gamma \left\{ \int_{t_0-R^2}^{t_0} [\text{meas } A^-_{k,R}(\tau)]^{\frac{r}{q}}\, d\tau \right\}^{\frac{2}{r}(1+\kappa)} +$$

$$+ 2\nu(\sigma_2 R^2)^{-1} \int\!\!\int_{Q_R} (v-k)^- \, dx d\tau \quad .$$

Inequalities (3.5) hold for all $\mu^- \le k \le \mu^- + \frac{\omega}{2^s}$, all $\sigma_1, \sigma_2 \in (0,1)$ and all cylinders Q_R, $0 < R \le R_0$.

Set

$$R_n = \frac{R_0}{2} + \frac{R_0}{2^{n+2}}$$

$$\bar{R}_n = \frac{R_0}{2} + \frac{3R_0}{2^{n+4}} \quad ,$$

and consider the cylinders Q_{R_n} and

$$\bar{Q}_n \equiv \{|x-x_0| < \bar{R}_n\} \times \{t_0 - R_n^2, t_0\}$$

$$\bar{\bar{Q}}_n \equiv \{|x-x_0| < \bar{R}_n\} \times \{t_0 - R_{n+1}^2, t_0\} \quad .$$

Obviously

$$Q_{R_{n+1}} \subset \bar{\bar{Q}}_n \subset \bar{Q}_n \subset Q_{R_n} \quad .$$

Construct smooth cutoff functions $x \to \zeta_n(x)$ as follows

(i) $\zeta_n(x) \equiv 1 \qquad |x - x_0| < R_{n+1}$

(ii) $\zeta_n(x) = 0 \qquad |x - x_0| > \frac{1}{2}\,[R_n + R_{n+1}] = \bar{R}_n$

(iii) $|\nabla\zeta_n(x)| \leq 2^{n+4}/R_0$.

For simplicity of notation set

$$k_1 = \mu^- + \frac{\omega}{2^s} \quad .$$

Our purpose is to apply (3.5) to the pair of cylinders Q_{R_n} and $\bar{\bar{Q}}_n$ for the decreasing levels

$$k_n = (k_1 - \frac{1}{2}H) + \frac{1}{2^n}H \quad , \quad n = 1, 2, \ldots ,$$

which as easily verified satisfy

$$\mu^- \leq k_n \leq k_1 \quad .$$

Set

$$y_n = \int_{Q_{R_n}}\!\!\int [(v - k_n)^-]^2\, dx d\tau \quad \text{and}$$

$$z_n = \left\{ \int_{t_0 - R_n^2}^{t_0} [\text{meas } A^-_{k_n, R_n}(\tau)]^{\frac{r}{q}}\, d\tau \right\}^{\frac{2}{r}} \quad .$$

As $n \to \infty$

$$y_n \to y_\infty = \int_{Q_{\frac{R_0}{2}}} [(v - (k_1 - \tfrac{1}{2} H))^-]^2 \, dx d\tau$$

$$z_n \to z_\infty = \left\{ \int_{t_0 - \frac{R_0^2}{4}}^{t_0} [\text{meas } A^-_{k_1 - \frac{1}{2} H, R_0/2}(\tau)]^{\frac{r}{q}} \, d\tau \right\}^{\frac{2}{r}} .$$

Therefore the lemma will be proved if we can show that $y_\infty = z_\infty = 0$.

Claim: The numbers

$$Y_n = \frac{y_n}{H^2 R_0^{N+2}} , \quad Z_n = \frac{z_n}{R_0^N} ,$$

satisfy the recursion inequalities

$$\text{[I]} \qquad Y_{n+1} \le \frac{\tilde{C} 2^{5n}}{H} \left[Y_n^{1+\frac{2}{N+2}} + Y_n^{\frac{2}{N+2}} Z_n^{1+\kappa} \right]$$

$$\text{[II]} \qquad Z_{n+1} \le \frac{\tilde{C} 2^{5n}}{H} \left[Y_n + Z_n^{1+\kappa} \right] ,$$

where

$$\tilde{C} = 2^{12} \beta^2 \gamma_0 .$$

Here β is the constant appearing in the embedding lemma 2.4, and $\gamma_0 = \max\{\nu, (1 + \gamma)(2M + \delta)\}$.

We remark that $\tilde{C}$ depends only upon the data and the dimension N.

Proof of the claim: We use here the method of [18] page 106. Set

$$P_n = \int_{t_0-R_{n+1}^2}^{t_0} [\text{meas } A^-_{k_{n+1},\bar{R}_{n+1}}(\tau)]\,d\tau \quad ,$$

and observe that

$$P_n \le (k_n - k_{n+1})^{-2}\, y_n = \left(\frac{2^{n+1}}{H}\right)^2 y_n \quad .$$

By virtue of lemma 2.4 applied over the cylinder $\bar{\bar{Q}}_n$ we have

$$\text{(3.6)} \qquad y_{n+1} \le \int\!\!\int_{\bar{\bar{Q}}_n} [(v - k_{n+1})]^{-2} \zeta_n^2(x)\,dx\,d\tau \le$$

$$\le \beta^2 P_n^{\frac{2}{N+2}} \left| (v - k_{n+1})^- \zeta_n \right|^2_{V_2^{1,0}(\bar{\bar{Q}}_n)} \quad .$$

Estimate of $\left| (v - k_{n+1})^- \zeta_n \right|^2_{V_2^{1,0}(\bar{\bar{Q}}_n)}$:

$$\left| (v - k_{n+1})^- \zeta_n \right|^2_{V_2^{1,0}(\bar{\bar{Q}}_n)} \le \left| (v - k_{n+1})^- \right|^2_{V_2^{1,0}(\bar{\bar{Q}}_{R_n})} +$$

$$+ 2 \int\!\!\int_{\bar{\bar{Q}}_n} [(v - k_{n+1})^-]^2 |\nabla \zeta_n(x)|\,dx\,d\tau = J_n^{(1)} + J_n^{(2)} \quad .$$

For $J_n^{(2)}$ we have

$$J_n^{(2)} \le 4R_0^{-1} 2^{n+4} \int\!\!\int_{Q_{R_N}} \{[(v - k_n)^-]^2 + (k_n - k_{n+1})^2\} \chi(v < k_{n+1})\,dx\,d\tau \le$$

$$\le 8\; 2^{n+4}\, y_n \quad .$$

In order to estimate $J_n^{(1)}$ we use inequalities (3.5) for the pair of cylinders $\bar{\bar{Q}}_n$ and Q_{R_n}. Notice that in this connection

$$(\sigma_1 R_n)^{-2} = R_0^{-2}\, 2^{2(n+4)} \quad ; \quad (\sigma_2 R_n^2)^{-1} < R_0^{-2}\, 2^{n+3} \quad ,$$

so that from (3.5) we deduce

$$J_n^{(1)} \le \frac{2\gamma}{R_0^2}\, 2^{2(n+4)} \int\!\!\int_{Q_{R_n}} [(v - k_{n+1})^-]^2 dx d\tau + \gamma\, z_n^{1+\kappa} +$$

$$+ \frac{\nu 2^{n+4}}{R_0^2} \int\!\!\int_{Q_{R_n}} (v - k_{n+1})^- \, dx d\tau \le$$

$$\le \frac{8\gamma}{R_0^2}\, 2^{2(n+4)}\, y_n + \gamma\, z_n^{1+\kappa} + \frac{\nu}{R_0^2}\, 2^{n+4}\, H \int_{t_0 - R_n^2}^{t_0} [\text{meas } A^-_{k_{n+1}, R_n}(\tau)] d\tau \quad .$$

Since

$$\int_{t_0 - R_0^2}^{t_0} [\text{meas } A^-_{k_{n+1}, R_n}(\tau)] d\tau \le (k_n - k_{n+1})^{-2}\, y_n = \frac{(2^{n+1})^2}{H^2}\, y_n \quad , \quad \text{and}$$

since $H \le 2M + \delta$, setting $\gamma_0 = \max\{\nu, (1+\gamma)(2M+\delta)\}$ above yields

$$J_n^{(1)} \le \frac{8(1+2^4)\gamma_0 2^{3(n+1)}}{R_0^2 H} \{y_n + R_0^2\, z_n^{1+\kappa}\} \quad .$$

Combining the estimates for $J_n^{(1)}$ and $J_n^{(2)}$ we have

$$(3.7) \qquad \| (v - k_{n+1})^- \zeta_n \|^2_{V_2^{1,0}(\bar{\bar{Q}}_n)} \le \frac{2^8 \gamma_0 (2^{n+1})^3}{R_0^2 H} \{y_n + R_0^2\, z_n^{1+\kappa}\}$$

<u>Estimate of Y_{n+1}:</u> We carry (3.7) in (3.6) and employ the estimate of P_n to obtain

$$\frac{Y_{n+1}}{H^2} \le \frac{2^8\beta^2\gamma_0(2^{n+1})^{3+\frac{4}{N+2}}}{R_0^2 H} \left\{ \left(\frac{Y_n}{H^2} \right)^{1+\frac{2}{N+2}} + \left(\frac{Y_n}{H^2} \right)^{\frac{2}{N+2}} \frac{R_0^2 \; Z_n^{1+\kappa}}{H^2} \right\}$$

and dividing by R_0^{N+2}

$$Y_{n+1} \le \frac{\tilde{C}2^{5n}}{H} \left\{ Y_n^{1+\frac{2}{N+2}} + Y_n^{\frac{2}{N+2}} Z_n^{1+\kappa} \frac{R_0^{N\kappa}}{H^2} \right\}$$

Now if $H^2 < R_0^{N\kappa}$, $\frac{R_0^{N\kappa}}{H^2} < 1$ and [I] is proved.

To prove [II] observe that

$$z_{n+1}(k_n - k_{n+1})^2 = (k_n - k_{n+1})^2 \left\{ \int_{t_0-R_{n+1}^2}^{t_0} [\text{meas } A^-_{k_{n+1},R_{n+1}}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}} \le$$

$$\le \| (v - k_n)^- \zeta_n \|^2_{q,r,\bar{\bar{Q}}_n} \le \text{ by the embedding lemma 2.4 } \le$$

$$\le \beta^2 |(v - k_n)^- \zeta_n|^2_{V_2^{1,0}(\bar{\bar{Q}}_n)} .$$

The last term is estimated in (3.7) so that [II] follow at once.

<u>Proof of lemma 3.1 concluded</u>:

By lemma 5.7 of [18] page 96, there exists a number $\lambda > 0$ such that if

$$Y_1 < \lambda \;\; ; \;\; Z_1 < \lambda^{\frac{1}{1+\kappa}} ,$$

then the recursion inequalities [I] - [II] imply that $Y_n, Z_n \to 0$ as $n \to \infty$.
From [18] setting

$$d = \min\left\{ \frac{2}{N+2} \, ; \, \frac{\kappa}{1+\kappa} \right\} ,$$

the number λ is given by

$$\lambda = \min\left\{ \left(\frac{H}{2\tilde{C}} \right)^{\frac{N+2}{2}} 2^{-\frac{5(N+2)}{2d}} \, ; \, \left(\frac{H}{2\tilde{C}} \right)^{\frac{1+\kappa}{\kappa}} 2^{-\frac{5}{\kappa d}} \right\} .$$

Now since $\kappa = \frac{2\kappa_1}{N}$, $\kappa_1 \in (0,1)$ and $\frac{H}{\tilde{C}} < 1$, above gives

$$d = \frac{\kappa}{1+\kappa} \, ; \quad \lambda \geq \lambda_0 = \left(\frac{1}{2^{s+2}\tilde{C}} \right)^{\frac{1+\kappa}{\kappa}} \min\left\{ 2^{\frac{-5(N+2)}{2d}} \, , \, 2^{\frac{-5}{\kappa d}} \right\} \cdot \omega^{\frac{1+\kappa}{\kappa}} .$$

Where ( ) has been used. Set:

$$\kappa_n c_0 = \left(\frac{1}{2^{s+2}\tilde{C}} \right)^{\frac{1+\kappa}{\kappa}} \min\left\{ 2^{\frac{-5(N+2)}{2d}} \, , \, 2^{\frac{-5}{\kappa d}} \right\}$$

and notice that c_0 depends only upon the data and not upon R_0 nor ω.

The lemma follows if $Y_1 \leq \kappa_n c_0 \, \omega^{\frac{2\kappa_1}{2\kappa_1+N}}$, i.e. if

$$Y_1 = \frac{y_1}{H^2 R_0^{N+2}} = \frac{1}{H^2 R_0^{N+2}} \int\!\!\int_{Q_{R_0}} |(v - (\mu^- + \frac{\omega}{2^s}))^-|^2 dx d\tau \leq$$

$$\leq \frac{1}{R_0^{N+2}} \operatorname{meas} Q^-_{R_0}(\mu^- + \frac{\omega}{2^s}) = \frac{1}{R_0^{N+2}} c_0 \, \omega^{\frac{2\kappa_1+N}{2\kappa_1}} \kappa_N h_0^{N+2} = \nu_0 .$$

From now on, for simplicity of notation we set

$$b = \frac{2\kappa_1 + N}{2\kappa_1} , \quad \nu_0 = c_0 \, \omega^b$$

and remark that b depends only upon the data and not upon ω nor R_0.

Remark: By selecting in inequalities (2.7) the constant γ large enough, we see that in [I] - [II] the constant $\tilde{C}$ can be made as large as we please so that without loss of generality we might assume that

$$c_0 \omega^b \le \frac{1}{2} \ .$$

Suppose now that the assumption of lemma 3.1 fails; then since

$$\mu^+ - \frac{\omega}{2^s} > \left|\frac{\omega}{2^s} + \mu^-\right|$$

$$\operatorname{meas} Q^+_{R_0}(\mu^+ - \frac{\omega}{2^s}) \le \kappa_N R_0^{N+2} - \theta_0 \kappa_N R_0^{N+2} \ .$$

$$= (1 - \theta_0)\kappa_N R_0^{N+2} \ .$$

Lemma 3.2: Suppose that $k \in \mathbb{R}^+$ and that

$$\operatorname{meas} Q^+_{R_0}(k) \le (1 - \theta_0)\kappa_N R_0^{N+2} \ ,$$

then for every $\alpha \in (0,\theta_0)$, there exists

$$\tau \in [t_0 - R_0^2, t_0 - \alpha R_0^2] \ ,$$

such that

$$\operatorname{meas} A^+_{k,R_0}(\tau) \le \frac{1 - \theta_0}{1 - \alpha} \kappa_N R_0^N \ .$$

Proof of lemma 3.2: If not, for all $\tau \in [t_0 - R_0^2, t_0 - \alpha R_0^2]$

$$\operatorname{meas} A^+_{k,R_0}(\tau) > \frac{1-\theta_0}{1-\alpha} \kappa_N R_0^N \quad \text{and}$$

$$\operatorname{meas} Q^+_{R_0}(k) \geq \int_{t_0 - R_0^2}^{t_0 - \alpha R_0^2} \operatorname{meas} A^+_{k,R_0}(\tau)\, d\tau >$$

$$> (1-\theta_0)\kappa_N R_0^{N+2} \quad .$$

We will choose

$$\alpha = \frac{\theta_0}{2} = \frac{c_0 \omega^b}{2} \quad ,$$

and use the previous lemma for the levels $k = \mu^+ - \frac{\omega}{2^p}$, $\forall p \geq s$.

Lemma 3.3: Let $\alpha = \frac{\theta_0}{2} = \frac{c_0\omega^b}{2}$ and consider the cylinder

$$Q^\alpha_{R_0} \equiv \{|x - x_0| < R_0\} \times [t_0 - \alpha R_0^2, t_0] \quad .$$

There exists $p_0 \in \mathbb{N}$ dependent upon ω (and hence α) such that if

$$\frac{\omega}{2^{p_0}} \geq R_0^{N\kappa/2} \quad , \quad \text{then}$$

$$\operatorname{meas} A^+_{\mu^+ - \frac{\omega}{2^{p_0}},\, R_0}(t) < \left[1 - \left(\frac{\theta_0}{2}\right)^2\right] \kappa_N R_0^N \quad ,$$

for all $t \in [t_0 - \alpha R_0^2, t_0]$.

Proof of lemma 3.3: Consider lemma 2.2 applied to the function

$(x,t) \to (v - (\mu^+ - \frac{\omega}{2^s}))^+ (x,t)$, in the cylinder

$$Q_{R_0}^{\tau} \equiv \{|x - x_0| < R_0\} \times [\tau, t_0] \quad ,$$

for $\eta = \frac{\omega}{2^p}$, $p \geq s + 2$. Here $t_0 - R_0^2 \leq \tau \leq t_0 - \alpha R_0^2$ is the number claimed in lemma 3.2. Notice that

$$\operatorname*{ess\,sup}_{Q_{R_0}^{\tau}} (v - (\mu^+ - \frac{\omega}{2^s}))^+ \leq \frac{\omega}{2^s} \quad ,$$

therefore for all $t \in [\tau, t_0]$ lemma 2.2 gives

$$(3.8) \qquad \int_{B(R_0 - \sigma_1 R_0)} \ln_+^2 \left[\frac{\frac{\omega}{2^s}}{\frac{\omega}{2^s} - (v - (\mu^+ - \frac{\omega}{2^s}))^+ + \frac{\omega}{2^p}} \right] (x,t)dx \leq$$

$$\leq \int_{B(R_0)} \ln_+^2 \left[\frac{\frac{\omega}{2^s}}{\frac{\omega}{2^s} - (v - (\mu^+ - \frac{\omega}{2^s}))^+ + \frac{\omega}{2^p}} \right] (x,\tau)dx +$$

$$+ \frac{C}{\sigma_1^2} \left(1 + \ln \frac{\frac{\omega}{2^s}}{\frac{\omega}{2^p}} \right) \left(1 + \frac{R_0^{N\kappa}}{\left(\frac{\omega}{2^p}\right)^2} \right) \kappa_N R_0^N \quad .$$

Let $p_0 > p$ to be selected, then if $\frac{\omega}{2^{p_0}} \geq R_0^{\frac{N\kappa}{2}}$, the last term is majorized by

$$\frac{4C}{\sigma_1^2} \ln 2^{p-s} \kappa_N R_0^N \quad , \quad p \geq s + 2 \quad .$$

We estimate the remaining terms in (3.8) as follows.

$$\int_{B(R_0)} \ell n^{+2}\left[\frac{\frac{\omega}{2^s}}{\frac{\omega}{2^s} - (v - (\mu^+ - \frac{\omega}{2^s}))^+ + \frac{\omega}{2^p}}\right](x,t)\,dx \leq$$

$$\leq [\ell n\ 2^{p-s}]^2 \text{ meas } A^+_{\mu^+ - \frac{\omega}{2^s}, R_0}(\tau) \leq \text{ by lemma 3.2 } \leq$$

$$\leq [\ell n\ 2^{p-s}]^2 \left(\frac{1-\theta_0}{1-\alpha}\right) \kappa_N R_0^N \quad .$$

For the left hand side we have

$$\int_{B(R_0-\sigma_1 R_0)} \ell n^{+2}\left[\frac{\frac{\omega}{2^s}}{\frac{\omega}{2^s} - (v - (\mu^+ - \frac{\omega}{2^s}))^+ + \frac{\omega}{2^p}}\right](x,t)\,dx \geq$$

$$\geq \int_{B(R_0-\sigma_1 R_0)\cap[v>\mu^+ - \frac{\omega}{2^p}]} \ell n^{+2}\left[\frac{\frac{\omega}{2^s}}{\frac{\omega}{2^s} - (v - (\mu^+ - \frac{\omega}{2^s}))^+ + \frac{\omega}{2^p}}\right](x,t)\,dx \geq$$

$$\geq \left[\ell n\left(\frac{\frac{\omega}{2^s}}{2\frac{\omega}{2^p}}\right)\right]^2 \text{ meas } A^+_{\mu^+ - \frac{\omega}{2^p}, R_0-\sigma_1 R_0}(t) \quad .$$

These estimates in (3.8) give the inequality

$$\text{(3.9)} \qquad \text{meas } A^+_{\mu^+ - \frac{\omega}{2^p}, R_0-\sigma_1 R_0}(t) \leq \left(\frac{\ell n\ 2^{p-s}}{\ell n\ 2^{p-s-1}}\right)^2 \left(\frac{1-\theta_0}{1-\alpha}\right) \kappa_N R_0^N +$$

$$+ \frac{4C}{\sigma_1^2} \frac{\ell n\, 2^{p-s}}{(\ell n\, 2^{p-s-1})^2} \kappa_N R_0^N =$$

$$= \left(\frac{p - s}{p - s - 1}\right)^2 \left(\frac{1 - \theta_0}{1 - \alpha}\right) \kappa_N R_0^N + \frac{4C}{\sigma_1^2 \,\ell n\, 2} \frac{p - s}{(p - s - 1)^2} \kappa_N R_0^N .$$

Now

$$\text{meas } A^+_{\mu^+ - \frac{\omega}{2^p}, R_0}(t) \leq \text{meas } A^+_{\mu^+ - \frac{\omega}{2^p}, R_0 - \sigma_1 R_0}(t) +$$

$$+ \text{meas } [B(R_0) \backslash B(R_0 - \sigma_1 R_0)] \leq \text{meas } A^+_{\mu^+ - \frac{\omega}{2^p}, R_0 - \sigma_1 R_0}(t) +$$

$$+ N\sigma_1 \kappa_N R_0^N ,$$

therefore by virtue of (3.9)

$$\text{meas } A^+_{\mu^+ - \frac{\omega}{2^p}, R_0}(t) \leq \left\{ \left(\frac{p - s}{p - s - 1}\right)^2 \left(\frac{1 - \theta_0}{1 - \alpha}\right) + \frac{4C}{\sigma_1^2 \,\ell n\, 2} \frac{p - s}{(p - s - 1)^2} + \right.$$

$$\left. + N\sigma_1 \right\} \kappa_N R_0^N .$$

This inequality holds for all $\sigma_1 \in (0,1)$, all $p > s + 2$ and all $t \in [\tau, t_0]$.

Select $\sigma_1 = \frac{3}{8} \frac{\theta_0^2}{N}$, and p_0 so large that

$$\frac{4C}{\sigma_1^2 \,\ell n\, 2} \frac{p_0 - s}{(p_0 - s - 1)^2} < \frac{3\theta_0^2}{8} , \quad \text{and}$$

$$\left(\frac{p_0 - s}{p_0 - s - 1}\right)^2 \leq (1 - \alpha)(1 + \theta_0) ,$$

to obtain

$$\text{meas}\, A^{+}_{\mu^+ - \frac{\omega}{2^{p_0}}, R_0}(t) \leq \left[1 - \left(\frac{\theta_0}{2} \right)^2 \right] \kappa_N R_0^N \ .$$

$$= \left[1 - \left(\frac{c_0 \omega^b}{2} \right)^2 \right] \kappa_N R_0^N \ .$$

This proves the lemma.

Remark: It is easily seen that a suitable choice of p_0 is

$$p_0 = s + 3 + \left[\frac{C_1}{(c_0{}^b)^6} \right] , \tag{3.10}$$

where [a] denotes the largest integer contained in a, and

$$C_1 = \frac{2^8 N^2 C}{\ell n\ 2}$$

Notice that C_1 depends only upon the data and not upon ω nor R_0.

Remark: Since for $q \geq p_0$, $A^{+}_{\mu^+ - \frac{\omega}{2^q}, R_0}(t) \subset A^{+}_{\mu^+ - \frac{\omega}{2^{p_0}}, R_0}(t)$, we have that

$$\text{meas}\, A^{+}_{\mu^+ - \frac{\omega}{2^q}, R_0}(t) \leq \left[1 - \left(\frac{\theta_0}{2} \right)^2 \right] \kappa_N R_0^N \ , \quad \forall q \geq p_0 \ ,$$

and for all $t \in [t_0 - \alpha R_0^2, t_0]$.

The subsequent arguments will be carried over the cylinder $Q^{\alpha}_{R_0}$.

For $k > 0$ we also denote

$$Q^{\alpha}_{R_0}(k) = \{(x,t) \in Q^{\alpha}_{R_0} \mid v(x,t) > k\} \ .$$

Lemma 3.4: For every $\theta_1 > 0$, there exists $q_0 \in \mathbb{N}$, $q_0 > p_0$ such that if

$$\frac{\omega}{2^{q_0}} > R_0^{\frac{N\kappa}{2}} \quad , \quad \text{then}$$

$$\text{meas } Q^{\alpha}_{R_0}\left(\mu^+ - \frac{\omega}{2^{q_0}}\right) \le \theta_1 \kappa_N R_0^{N+2} \quad .$$

Proof of lemma 3.4: Lemma 3.3 and the remarks following it imply that

$$\text{meas}\{B(R_0)\setminus A^+_{\mu^+ - \frac{\omega}{2^q}, R_0}(t)\} \ge \left(\frac{\theta_0}{2}\right)^2 \kappa_N R_0^N \quad , \quad \forall q \ge p_0$$

for all $t \in [t_0 - \alpha R_0^2, t_0]$.

Apply inequality (2.9) to the function $x \to v(x,t)$ in the ball $B(R_0)\times\{t\}$ for the levels

$$\ell = \mu^+ - \frac{\omega}{2^{q+1}} \quad , \quad k = \mu^+ - \frac{\omega}{2^q} \quad , \quad q_0 > q > p_0 \quad ,$$

where q_0 has to be chosen. If we do this for all $t \in [t_0 - \alpha R_0^2, t_0]$ we obtain

$$\left(\frac{\omega}{2^{q+1}}\right) \text{meas}\{A^+_{\mu^+ - \frac{\omega}{2^{q+1}}, R_0}(t)\} \le D \frac{R_0^{N+1}}{\text{meas}[B(R_0)\setminus A^+_{\mu^+ - \frac{\omega}{2^q}, R_0}(t)]} \quad .$$

$$\int_{A^+_{k,R_0}(t)\setminus A^+_{\ell,R_0}(t)} |\nabla_x v|\, dx \le \frac{4DR_0}{\kappa_N \theta_0^2} \int_{A^+_{k,R_0}(t)\setminus A^+_{\ell,R_0}(t)} |\nabla_x v|\, dx \quad .$$

Integrate both the sides of this inequality over $[t_0 - \alpha R_0^2, t_0]$, square and use Hölder's inequality on the right side, to obtain

$$(3.11)\quad \left(\frac{\omega}{2^{q+1}}\right)^2 \left[\text{meas}\; Q^{\alpha}_{R_0}\left(\mu^+ - \frac{\omega}{2^{q+1}}\right)\right]^2 \leq \left[\frac{4D}{\theta_0^2 \kappa_N}\right]^2 R_0^2 \;.$$

$$\left[\int_{t_0-\alpha R_0^2}^{t_0} \int_{A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)} |\nabla_x v|^2\, dx d\tau\right]\left[\int_{t_0-\alpha R_0^2}^{t_0} [\text{meas}\; A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)]d\tau\right]$$

$$\leq \left[\frac{4D}{\theta_0^2 \kappa_N}\right]^2 R_0^2 \left|(v - (\mu^+ - \frac{\omega}{2^q}))^+\right|^2_{V_2^{1,0}(Q_{R_0})}$$

$$\left[\int_{t_0-\alpha R_0^2}^{t_0} [\text{meas}\; A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)]d\tau\right] \;.$$

In order to estimate the $V_2^{1,0}(Q_{R_0})$-norm of $(v - (\mu^+ - \frac{\omega}{2^q}))^+$ we use inequalities (2.7) applied to the pair of cylinders Q_{R_0}, Q_{2R_0}. Notice that in this connection $\underset{Q_{2R_0}}{\text{ess sup}}\,(v - (\mu^+ - \frac{\omega}{2^q}))^+ \leq \frac{\omega}{2^q} < \delta$ and that

$$(\sigma_1 R_0)^{-2} = 4R_0^{-2} \;;\; (\sigma_2 R_0^2)^{-1} = \frac{4}{3} R_0^{-2} \;.$$

Moreover observe that since $\mu^+ \geq |\mu^-|$, we have

$$\mu^+ - \frac{\omega}{2^q} > 0 \quad ,$$

so that $\phi^+_a(\hat{\mu} - \frac{\omega_0}{2^q}, t_0 - 4R_0^2, t) = 0$, $t \in [t_0-4R_0^2, t_0]$. Inequalities (2.7) now give

$$\left|(v - (\mu^+ - \frac{\omega}{2^q}))^+\right|^2_{V_2^{1,0}(Q_{R_0})} \leq \gamma(4 + \frac{4}{3})R_0^{-2} \left\|(v - (\mu^+ - \frac{\omega}{2^q}))^+\right\|^2_{2,Q_{2R_0}} +$$

$$+\gamma\left\{\int_{t_0-4R_0^2}^{t_0}\left[\text{meas } A^+_{\mu^+-\frac{\omega}{2^q},2R_0}(\tau)\right]^{\frac{r}{q}}d\tau\right\}^{\frac{2}{r}(1+\kappa)} \leq$$

$$\leq \gamma\,\frac{2^{N+6}}{3}\left(\frac{\omega}{2^q}\right)^2\kappa_N R_0^N+\gamma 2^{N(1+\kappa)}\kappa_N^{\frac{2}{q}(1+\kappa)-1}R_0^{N\kappa}\,\kappa_N R_0^N\ ,$$

where (2.4) has been used. By assumption

$$R_0^{N\kappa}\leq\left(\frac{\omega}{2^{q_0}}\right)^2<\left(\frac{\omega}{2^q}\right)^2\ ,$$

so that there exists a constant C_2 depending only upon the dimension N and the data, such that

$$\left|(v-(\mu^+-\frac{\omega}{2^q}))^+\right|^2_{V_2^{1,0}(Q_{R_0})}\leq C_2\left(\frac{\omega}{2^q}\right)^2\kappa_N R_0^N\ .$$

Carrying this in (3.11) and dividing by $\left(\frac{\omega}{2^{q+1}}\right)^2$, gives

(3.12) $$[\text{meas } Q^\alpha_{R_0}(\mu^+-\frac{\omega}{2^{q+1}})]^2\leq 4C_2\left[\frac{4D}{\kappa_N}\right]^2\frac{1}{\theta_0^4}\kappa_N R_0^{N+2}\ .$$

$$\left[\int_{t_0-\alpha R_0^2}^{t_0}[\text{meas } A^+_{k,R_0}(\tau)\backslash A^+_{\ell,R_0}(\tau)]d\tau\right]\ .$$

We add inequalities (3.12) with respect to q, from p_0 to q_0-1 and obtain

$$(q_0-p_0)[\text{meas } Q_{R_0'}(\mu^+-\frac{\omega}{2^{q_0}})]^2\leq 4C_2\left[\frac{4D}{\kappa_N}\right]^2\frac{1}{\theta_0^4}\kappa_N R_0^{N+2}\ .$$

$$\sum_{q=p_0}^{q_0-1} \int_{t_0-\alpha R_0^2}^{t_0} [\text{meas } A^+_{\mu^+ - \frac{\omega}{2^q}, R_0}(\tau) \backslash A^+_{\mu^+ - \frac{\omega}{2^{q+1}}, R_0}(\tau)] d\tau \leq$$

$$\leq 4C_2 \left[\frac{4D}{\kappa_N}\right]^2 \frac{1}{\theta_0^4} \alpha (\kappa_N R_0^{N+2})^2 \quad .$$

Now recall that $\alpha = \frac{\theta_0}{2}$ set

$$C_3 = 2C_2 \left[\frac{4D}{\kappa_N}\right]^2$$

and observe that C_3 depends only upon the data and the dimension N.

Dividing the inequality above by $q_0 - p_0$, to prove the lemma we have only to choose q_0 so large that

$$\frac{1}{q_0 - p_0} \frac{C_3}{\theta_0^3} \leq \theta_1^2 \quad .$$

We will select

$$q_0 = p_0 + 1 + \left[\frac{C_3}{\theta_0^3 \theta_1^2} \right] \tag{3.13}$$

Remark: The proof of lemma 3.4 is an adaptation of a similar result of [18], namely lemma 7.2 page 114.

Consider now the pair of cylinders $Q^{\alpha}_{R_0}$ and

$$Q^{\alpha}_{\frac{R_0}{2}} = \{ |x - x_0| < \frac{R_0}{2} \} \times [t_0 - \alpha \frac{R_0^2}{4}, t_0] \quad .$$

For them we have the following result

Lemma 3.5: There exists a number $\theta_1 > 0$ depending upon α, N and the data, such that if

$$\operatorname{meas} Q^{\alpha}_{R_0}\left(\mu^+ - \frac{\omega}{2^{q_0}}\right) < \theta_1 \kappa_N R_0^{N+2} ,$$

then either

$$\text{(i)}\quad H = \operatorname*{ess\,sup}_{Q^{\alpha}_{R_0}} \left(v - \left(\mu^+ - \frac{\omega}{2^{q_0}}\right)\right)^+ \le R_0^{\frac{N\kappa}{2}} , \quad \text{or}$$

$$\text{(ii)}\quad \operatorname{meas} Q^{\alpha}_{\frac{R_0}{2}}\left(\mu^+ - \frac{\omega}{2^{q_0}} + \frac{1}{2} H\right) = 0 .$$

Proof of lemma 3.5: The proof is very similar to the *proof of lemma* 3.1. We reproduce the main steps mainly to trace the dependence of q_0 on θ_0 (and hence on ω). Let R_n, $\bar{R}_n$ be defined as before, and consider the cylinders

$$Q^{\alpha}_{R_n} \equiv \{|x - x_0| < R_n\} \times [t_0 - \alpha R_n^2, t_0]$$

$$\bar{Q}^{\alpha}_n \equiv \{|x - x_0| < \bar{R}_n\} \times [t_0 - \alpha R_n^2, t_0]$$

$$\bar{\bar{Q}}^{\alpha}_n \equiv \{|x - x_0| < \bar{R}_n\} \times [t_0 - \alpha R_{n+1}^2, t_0] ,$$

which satisfy the inclusions

$$Q^{\alpha}_{R_{n+1}} \subset \bar{\bar{Q}}^{\alpha}_n \subset \bar{Q}^{\alpha}_n \subset Q^{\alpha}_{R_n} .$$

We use inequalities (2.7) over $\bar{\bar{Q}}^{\alpha}_n$ and $Q^{\alpha}_{R_n}$, for the functions $(x,t) \to (v - k_n)^+$ where

$$k_n = (k_1 + \frac{1}{2} H) - \frac{1}{2^n} H \quad , \quad n = 1, 2, \ldots$$

$$k_1 = \mu^+ - \frac{\omega}{2^{q_0}} \quad .$$

Since $k_n \geq k_1 > 0$ in (2.7) we have

$$\phi_a^+(k_n, t_0 - \alpha R_n^2, t_0) = 0 \quad .$$

Note that in this case $(\sigma_1 R_n)^{-2} = R_0^{-2}\, 2^{2(n+4)}$ and $(\sigma_2 \alpha R_n^2)^{-1} = R_0^{-2}\, \alpha^{-1}\, 2^{n+3}$.
We have to show that the numbers

$$Y_n = \frac{1}{H^2 R_0^{N+2}}\, y_n = \frac{1}{H^2 R_0^{N+2}} \int_{Q_{R_n}^\alpha} \int (v - k_n)^{+2}\, dx d\tau$$

$$Z_n = \frac{z_n}{R_0^N} = \frac{1}{R_0^N} \left\{ \int_{t_0 - \alpha R_n}^{t_0} [\text{meas } A^+_{k_n, R_n}(\tau)]^{\frac{r}{q}}\, d\tau \right\}^{\frac{2}{r}} \quad ,$$

tend to zero as $n \to \infty$. Proceeding exactly as in lemma 3.1 we see that Y_n and Z_n satisfy the recursion inequalities

$$Y_{n+1} \leq \frac{\hat{C} 2^{4n}}{\alpha} [Y_n^{1+\frac{2}{N+2}} + Y_n^{\frac{2}{N+2}} Z_n^{1+\kappa}]$$

$$Z_{n+1} \leq \frac{\hat{C} 2^{4n}}{\alpha} [Y_n + Z_n^{1+\kappa}] \quad .$$

The lemma follows if

$$Y_1 \leq R_0^{-(N+2)} \text{ meas } Q'_{R_0}(k_1) < \delta_1 \cdot N =$$

$$= \left(\frac{1}{2\tilde{C}}\right)^{\frac{1+\kappa}{\kappa}} \alpha^{\frac{1+\kappa}{\kappa}} \min\{2^{-4\frac{N+2}{2d}}, 2^{-4\frac{1}{\kappa d}}\} ,$$

i.e. if

$$\theta_1 = C_4\alpha^b \quad , \quad \text{where} \quad C_4 = \frac{1}{\kappa_N} (2\tilde{C})^{-\frac{1+\kappa}{\kappa}} \min\{2^{-4\frac{N+2}{2d}} ; 2^{-4\frac{1}{\kappa d}}\} .$$

Here $\tilde{C}$ and d are as in lemma 3.1.

From (3.13) and the remarks above it follows that the conclusion of lemma 3.5 holds true if we choose

$$q_0 \geq p_0 + 1 + \left[\frac{C_3}{(C_4\alpha^b)^2\theta_0^3}\right] .$$

We recall that $\theta_0 = c_0\omega^b$, $\alpha = \frac{\theta_0}{2}$, therefore taking in account (3.10) we might select

$$q_0 = s + 4 + \left[\left\{ C_1 + \frac{2^{2b}C_3}{C_4^2} \right\} \frac{1}{(c_0\omega^b)^{\max\{6,2b+3\}}} \right] .$$

Set

$$A = \left\{ C_1 + \frac{2^{2b}C_3}{C_4^2} \right\} \frac{1}{c_0^{\max\{6,2b+3\}}}$$

and

$$a = b \max\{6 ; 2b + 3\} ,$$

so that

(3.14) $$q_0 = s + 4 + [\frac{A}{\omega^a}] .$$

We remark that s, A, a depend only upon the data and not upon ω nor the size of R_0.

Proof of the proposition: Suppose that

$$(3.15) \qquad \frac{\omega}{2^s} \cdot \frac{\omega}{2^{q_0+1}} \geq \frac{\omega}{2^{s+4+A/\omega^a}} \cdot (2R_0)^{\frac{N\kappa}{2}} .$$

Obviously either

$$1. \quad \text{meas}\, Q^-_{R_0}\left(\mu^- + \frac{\omega}{2^s}\right) \leq c_0\, \omega^{\frac{2\kappa_1}{N+2\kappa_1}}\, \kappa_N\, R_0^{N+2}$$

or

$$2. \quad \text{meas}\, Q^-_{R_0}\left(\mu^- + \frac{\omega}{2^s}\right) > c_0\, \omega^{\frac{2\kappa_1}{N+2\kappa_1}}\, \kappa_N\, R_0^{N+2} .$$

Case 1: By lemma 3.1 either

$$1.\text{a.} \quad \operatorname*{ess\,sup}_{Q_{R_0}} \left(v - \left(\mu^- + \frac{\omega}{2^s}\right)\right)^- \leq R_0^{\frac{N\kappa}{2}} , \quad \text{or}$$

$$1.\text{b.} \quad \text{meas}\, Q^-_{\frac{R_0}{2}}\left(\mu^- + \frac{\omega}{2^s} - \frac{1}{2} H\right) = 0 .$$

If 1.a occurs then

$$-\operatorname*{ess\,inf}_{Q_{R_0}} v \leq -\mu^- - \frac{\omega}{2^s} + R_0^{\frac{N\kappa}{2}} \leq \text{ by } (3.15) \leq$$

$$\leq -\mu^- - \frac{\omega}{2^{s+5+A/\omega^a}} .$$

Adding $\operatorname*{ess\,sup}_{Q_{R_0}} v$ on the left hand side and μ^+ on the right hand side we obtain

$$\operatorname*{ess\ osc}_{Q_{R_0}} v \le \operatorname*{ess\ osc}_{Q_{2R_0}} v - \frac{\omega}{2^{s+5+A/\omega^a}}$$

$$\le \omega\left(1 - \frac{1}{2^{s+5+A/\omega^a}}\right) .$$

If 1.b occurs then

$$-\operatorname*{ess\ inf}_{Q_{\frac{R_0}{2}}} v \le -\mu^- - \frac{\omega}{2^s} + \frac{1}{2}\left(-\mu^- + \frac{\omega}{2^s} + \mu^-\right)$$

$$= -\mu^- - \frac{\omega}{2^{s+1}}$$

i.e.

$$\operatorname*{ess\ osc}_{Q_{\frac{R_0}{2}}} v \le \omega\left(1 - \frac{1}{2^{s+5+A/\omega^a}}\right) .$$

Case 2: By lemmas 3.2 - 3.4, in view of (3.15), the assumptions of lemma 3.5 are verified. It gives the following alternative. Either

$$\text{2.a.}\quad \operatorname*{ess\ sup}_{Q'_{\frac{R_0}{2}}} v \le \mu^+ - \frac{\omega}{2^{q_0}} + R_0^{\frac{N\kappa}{2}} \le$$

$$\le \mu^+ - \frac{\omega}{2^{q_0+1}}$$

or

$$\text{2.b.}\quad \operatorname*{ess\ sup}_{Q'_{\frac{R_0}{2}}} v \le \mu^+ - \frac{\omega}{2^{q_0}} + \frac{\omega}{2^{q_0+1}} = \mu^+ - \frac{\omega}{2^{q_0+1}} .$$

Hence in either case

$$\operatorname*{ess\,osc}_{Q^{\omega}_{\frac{R_0}{2}}} v \le \omega\left(1 - \frac{1}{2^{s+5+A/\omega^a}}\right) \quad .$$

Now to determine R_* notice that by virtue of (3.15)

$$\omega\left(\frac{R_0}{2}\right)^2 = \frac{1}{2 \cdot 4^2}\, c_0 \omega^b \,(2R_0)^2 >$$

$$> 2^{sb-5}\, c_0 (2R_0)^{2+\frac{N\kappa b}{2}} \quad .$$

Setting $\delta_* = (2^{sb-5} c_0)^{\frac{1}{2}} \wedge 1$, and

$$h = 1 + \frac{N\kappa b}{4} \ ,$$

we have

$$\sqrt{\omega}\, \frac{R_0}{2} \ge \delta_* (2R_0)^h \equiv R_* \quad ,$$

so that $Q^{\omega}_{\frac{R_0}{2}} \supset Q_{R_*} \equiv \{|x - x_0| < R_*\} \times [t_0 - R_*^2,\ t_0]$.

It follows that

(3.16) $$\operatorname*{osc}_{Q_{R_*}} v \le \omega\left(1 - \frac{1}{2^{s_0+A/\omega^a}}\right)$$

where $s_0 = s + 5$. Finally if (3.3) is false then (3.16) follows at once. The proposition is proved.

4. <u>Proof of Theorem 1</u>:

We will prove the theorem by exploiting the results of the previous section. Proposition 3.1 is valid for any number ω satisfying

$$\operatorname*{ess\ osc}_{Q_{2R_0}} v \leq \omega \leq 2M \quad .$$

We stress the fact that the constants ξ_*, a, b, A, h in proposition 3.1 do not depend upon R_0 nor ω. Let $(x_0,t_0) \in \Omega_T$ be fixed and select $\omega = 2M \geq \operatorname*{osc}_{\Omega_T} v$. Let $0 < R_0 < \frac{1}{2}$ be so small that

$$Q_{2R_0} \subset \Omega_T$$

$$(4.1) \qquad (2R_0)^{\frac{N\kappa}{2}} \leq \frac{2M}{2^{s_0+A/(2M)^a}} \quad .$$

Define two sequences of positive real numbers $\{R_n\}$ and $\{M_n\}$ as follows

$$R_1 = 2R_0 \ ; \ R_{n+1} = \xi(R_n)^h \qquad n = 2, 3, \ldots$$

where $\xi = \min\{\xi_* \ ; \ 4^{-\frac{2a}{N\kappa}}\}$, and

$$M_1 = 2M \quad , \quad M_{n+1} = M_n\left(1 - \frac{1}{2^{s_0+A/M_n^a}}\right)$$

<u>Lemma 4.1</u>: $\{M_n\} \downarrow 0$, $\{R_n\} \downarrow 0$ and for all $n \in \mathbb{N}$

$$\operatorname*{ess\ osc}_{Q_{R_n}} v \leq M_n \quad .$$

Proof of Lemma 4.1: If $M_n > M_{n+1} > \dots > M_0 > 0$, then for all $n \in \mathbb{N}$

$$M_{n+1} \le M_n \varepsilon \;;\; \varepsilon = \left(1 - \frac{1}{2^{s_0+A/M_0^a}}\right) < 1 \quad .$$

Therefore

$$M_n \le M_0 \varepsilon^n \qquad n = 1, 2, \dots$$

which implies that $M_n \searrow 0$ as $n \to \infty$. A contradiction. The statement about $\{R_n\}$ is trivial.

In view of 4.1, Proposition 3.1 implies that

$$\operatorname*{ess\,osc}_{Q_{R_2}} v \le M_2 \quad .$$

Moreover

$$R_2^{\frac{N\kappa}{2}} = \xi^{\frac{N\kappa}{2}} \left(R_1^{\frac{N\kappa}{2}}\right)^h \le 4^{-a} \left(\frac{M_1}{2^{s_0+A/M_1^a}}\right)^h$$

For simplicity of notation set

$$\sigma(x) = 2^{s_0+A/x^a} \quad , \; x > 0 \; .$$

Then, using the definition of M_2,

$$R_2^{\frac{N\kappa}{2}} \le 4^{-a} \left(\frac{M_2}{\sigma(M_1)(1 - \sigma(M_1)^{-1})}\right)^h \le 4^{-a} \left(\frac{2M_2}{\sigma(M_1)}\right)^h =$$

$$= \frac{M_2}{\sigma(M_2)} \, \frac{\sigma(M_2)}{\sigma(M_1)} \, 4^{-a} \; 2 \left(\frac{2M_2}{\sigma(M_1)}\right)^{\frac{N\kappa b}{4}} \quad .$$

Recalling the definitions of b, s it is immediate to see that $2\left(\frac{2M_2}{\sigma(M_1)}\right)^{\frac{N\kappa b}{4}} < 1$.

Now for all $n \in \mathbb{N}$ it is easy to check that

$$\frac{\sigma(M_{n+1})}{\sigma(M_n)} \leq 4^a \quad ,$$

Hence

$$R_2^{\frac{N\kappa}{2}} \leq \frac{M_2}{2^{s_0 + A/M_2^a}} \quad .$$

We have shown that the two inequalities

$$\operatorname*{osc}_{Q_{R_1}} v \leq M_1$$

$$R_1^{\frac{N\kappa}{2}} \leq \frac{M_1}{2^{s_0 + A/M_1^a}}$$

imply the same two inequalities for R_2 and M_2. The same argument shows that if

$$\operatorname*{ess\,osc}_{Q_{R_n}} v \leq M_n$$

$$R_n^{\frac{N\kappa}{2}} \leq \frac{M_n}{2^{s_0 + A/M_n^a}} \quad ,$$

then the same inequalities are valid for $n + 1$. The lemma is proved.

As a consequence of lemma 4.1 we have that $\forall (x_0, t_0) \in \Omega_T$

$$\operatorname*{ess\,lim}_{(x,t) \to (x_0,t_0)} v(x,t)$$

exists. We define the function $(x,t) \to \tilde{v}(x,t)$ by setting

$$\tilde{v}(x_0,t_0) = \operatorname*{ess\,lim}_{(x,t)\to(x_0,t_0)} v(x,t) \quad .$$

Lemma 4.2: The function $(x,t) \to \tilde{v}(x,t)$ is a continuous representative out of the equivalence class v. Moreover if K is a compact contained in Ω_T there exists a nondecreasing continuous function $\omega_K(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+$, $\omega_K(0) = 0$ depending upon the data and dist$(K;\Gamma)$ such that

$$|\tilde{v}(x_1,t_1) - \tilde{v}(x_2,t_2)| \leq \omega_K(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

$$\forall (x_i,t_i) \in K \quad , \quad i = 1, 2 \quad .$$

The statement is a direct consequence of lemma 4.1 and establishes the interior regularity claimed by Theorem 1.

Remark: The continuity result is a consequence of inequalities (2.7) - (2.8) and lemma 2.2 solely.

5. Continuity up to the boundary:

Let $u \in W_2^{1,1}(\Omega_T)$ be a weak solution of (1.1) subject to certain boundary conditions. In this section we investigate under what circumstances the continuity of u can be extended to the closure of Ω_T. As remarked in the introduction, this is equivalent to prove the continuity of $(x,t) \to v(x,t)$ on the closure $\bar{\Omega}_T$.

Our study is divided in three parts. First we show the continuity of v at time $t = 0$. Then we investigate the regularity on the lateral part S_T of the parabolic boundary of Ω_T, for the cases of variational (Neumann) boundary conditions, and homogeneous Dirichlet boundary data. The method of proof in all three cases is similar to the one in section 3 and 4, and consists roughly speaking in constructing for every $(x_0,t_0) \in \bar{\Omega}_T$ a family of coaxial nested cylinders with same "vertex" (x_0,t_0) where the essential oscillation of v progressively decreases according to the rules imposed by the operator in (1.1) and the boundary data. Because of the information contained in the boundary data the analysis in the present situation is in fact simpler.

We will consider cylinders of two types.

Basis cylinders:

(BC) $(x_0,t_0) \in \overset{\circ}{\Omega}_T$; $Q_R \equiv \{|x - x_0| < R\} \times [t_0 - R^2, t_0]$

with $\{|x - x_0| < R\} \subset \Omega$ and $t_0 - R^2 < 0$.

Lateral cylinders:

(LC) $(x_0,t_0) \in S_T$; $Q_R \equiv \{|x - x_0| < R\} \times \{t_0 - R^2, t_0\}$.

The axis of (LC) lies on S_T and both (BC) and (LC) are not contained in Ω_T.

A - Continuity at $t = 0$

Let $v \in W_2^{1,1}(\Omega_T)$ satisfy identity (2.1) and in addition

$$v(x,0) = v_0(x) = \beta_0^{-1}(u_0(x)) ,$$

in the sense of the traces over Ω. Let the selection $w_0(x) \in \gamma(u_0(x))$ be given so that (2.1) is well defined for all $t_0 \geq 0$. We assume that $x \to v_0(x)$ is continuous in Ω, with modulus of continuity $s \to \omega_K(s)$ over a compact $K \subset \Omega$. Here $\omega_K(.)$ maps $\mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and $\omega_K(0) = 0$. Our task is to prove the following theorem.

Theorem 5.1: Let K be a compact of Ω. There exists a non-decreasing, continuous function $s \to \omega_K(s) : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_K(0) = 0$ such that

$$|v(x_1,t_1) - v(x_2,t_2)| \leq \omega_K(|x_1 - x_2| + |t_1 - t_2|^{1/2})$$

for all $(x_i,t_i) \in K' \times [0,T]$, $i = 1,2$, and every compact $K' \subset K$.

The function $\omega_K(.)$ depends only upon the data and the modulus of continuity of $v_0(x)$.

Clearly we have only to prove the continuity at $t = 0$, so that it will suffice only to consider cylinders (BC).

Lemma 5.1: Let $x_0 \in \Omega$ and $R > 0$ so small that $B(R) \subset \Omega$. Fix $0 < t_0 < R^2$ and over the cylinder (BC) Q_R consider the cut off function $x \to \zeta(x)$ selected as in (b). There exists a constant γ depending only upon the data such that the functions $(x,t) \to (v-k)^{\pm}(x,t)$ satisfy the inequalities

$$\|(v - k)^{\pm}\|^2_{2,B(R-\sigma_1 R)}(t) + \|\nabla_x (v - k)^{\pm}\|^2_{2,B(R-\sigma_1 R)\times(0,t)}$$

$$\leq \|(v - k)^{\pm}\|^2_{2,B(R)}(0) + \gamma[(\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}]\|(v - k)^{\pm}\|^2_{2,Q_R \cap \Omega_T} +$$

$$+ \gamma \left\{ \int_0^t [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\cdot)} +$$

$$+ \tilde{\Phi}^{\pm}_b(k,0,t) \; , \qquad \text{for all} \quad t \in [0,t_0] \; ,$$

provided that $k \in \mathbb{R}$ satisfies the restriction

$$\text{(5.2)} \qquad \operatorname*{ess\,sup}_{Q_R \cap \Omega_T} (v - k)^{\pm} < \delta \, .$$

Here δ is the same number introduced in (2.6) and

$$\tilde{\Phi}^{\pm}_b(k,0,t) = \mp \int_{\Omega} \nu(x,\tau)\chi[v \le 0](v-k)^{\pm} \zeta^2 \, dx \Big|_0^t$$

$$\pm \int_0^{t_0} \int_{\Omega} \nu(x,\tau)\chi[v \le 0] \frac{\partial}{\partial t}(v-k)^{\pm} \zeta^2(x) dx d\tau \; .$$

Inequalities (5.1) are derived in away similar to inequalities (2.7) - (2.8), the only difference being the domain of integration. In particular, the constant γ can be taken equal to the analogous constant appearing in (2.7) - (2.8).

Next we simplify (5.1) by imposing further restrictions on the levels k. Setting

$$k_1(R) = \sup_{B(R)} v_0(x) \; , \; k_2(R) = \inf_{B(R)} v_0(x) \; ,$$

for the oscillation of $x \to v_0(x)$ in $B(R)$ we have

$$\operatorname*{osc}_{B(R)} v_0(x) = k_1(R) - k_2(r) \equiv \omega_K(R) \; ,$$

for a compact K contained in Ω and containing $B(R)$. From now on we will keep fixed the compact K, and all the subsequent arguments will be carried over balls $B(R) \subset K$.

If in (5.1) we choose $k \geq k_1(R)$, then $(v - k)^+ (x,0) = 0$. Moreover if we look at v as extended over all Q_R in a way not to exceed $k_1(R)$, then $(v - k)^+$ is identically zero over that portion of Q_R not contained in Ω_T. Therefore if $k \geq k_1(R)$, the domains of integration in (5.1) can be replaced by $B(R - \sigma_1 R) \times [t_0 - R^2, t_0]$ on the left hand side and Q_R on the right hand side respectively. These remarks show that the function $(x,t) \to v(x,t)$ satisfies the inequalities

$$(5.1)^+ \qquad \|(v - k)^\pm\|^2_{V_2^{1,0}(B(R - \sigma_1 R) \times [t_0 - R^2, t_0])}$$

$$\leq \gamma[(\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}] \|(v - k)^+\|^2_{2,Q_R}$$

$$+ \gamma \left\{ \int_{t_0 - R^2}^{t_0} [\text{meas } A^+_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)}$$

$$+ \sup_{t \in [t_0 - R^2, t_0]} \tilde{\zeta}^+_b (k, t_0 - R^2, t)$$

for all $k \geq k_1(R)$ and satisfying (5.2).

A similar argument holds for $(x,t) \to (v - k)^-(x,t)$ provided that $k \leq k_2(R)$. It yields inequalities to which we will refer to as $(5.1)^-$.

We denote with s the smallest natural number satisfying

$$(5.3) \qquad \frac{2\gamma}{2^s} < 1 , \quad s \geq 2 .$$

Let ω be any positive number and construct the (BC)

$$Q^*_{R_0} \equiv \{|x - x_0| < R_0\} \times [t_0 - R_0^2, t_0] , \quad B(R_0) \subset K \subset \Omega$$

$$t_0 = \tau R_0^2 \quad , \quad \tau = \min\left\{ c_0 \omega^{\frac{2\kappa_1}{N+2\kappa_1}} ; \frac{1}{4} \right\} .$$

Here c_0 is the number claimed by Lemma 3.1.

Lemma 5.2: Assume that

(i) $2M \geq \omega \geq \operatorname*{osc}_{Q^*_{R_0} \cap \Omega_T} v$ (ii) $\frac{\omega}{2^{s+3}} > R_0^{\frac{N\kappa}{2}}$.

Then

$$\operatorname*{osc}_{Q^*_{\frac{R_0}{2}} \cap \Omega_T} v \leq \max\left\{ \omega - \frac{\omega}{2^{s+3}} ; 2^{s+1} \hat{\omega}_K(R_0) \right\} .$$

Proof of Lemma 5.2: If $\operatorname*{osc}_{Q^*_{R_0} \cap \Omega_T} v \leq \frac{\omega}{2^s}$ then the conclusion of the lemma is trivial. Analogously if $\omega \leq 2^{s+1} \hat{\omega}_K(R_0)$, there is nothing to prove. So assume that

(5.4) $$\operatorname*{osc}_{Q^*_{R_0} \cap \Omega_T} v > \frac{\omega}{2^s} > \frac{\omega}{2^{s+1}} > \hat{\omega}_K(R_0) .$$

Then at least one of the following two inequalities holds. Either

$(5.5)_1$ $$k_1(R_0) \leq \mu^+ - \frac{\omega}{2^{s+2}} , \quad \text{or}$$

$(5.5)_2$ $$k_2(R_0) \geq \mu^- + \frac{\omega}{2^{s+2}} .$$

Here

$$\mu^{+} = \operatorname*{ess\,sup}_{Q'_{R_0}\cap\Omega_T} v \quad ; \quad \mu^{-} = \operatorname*{ess\,inf}_{Q'_{R_0}\cap\Omega_T} v \; .$$

Indeed if both $(5.5)_1$ and $(5.5)_2$ are violated then

$$\omega_K(R_0) \geq k_1(R_0) - k_2(R_0) \geq \mu^{+} - \mu^{-} - \frac{\omega}{2^{s+1}} \geq$$

$$\geq \frac{\omega}{2^{s}} - \frac{\omega}{2^{s+1}} = \frac{\omega}{2^{s+1}} \; ,$$

contradicting (5.4).

Suppose that $(5.5)_2$ holds true and observe that

$$\operatorname{meas}\{(x,t) \in Q'_{R_0} \,|\, v(x,t) < \mu^{-} + \frac{\omega}{2^{s+2}} < k_2\} \leq$$

$$\leq \operatorname{meas}[Q'_{R_0} \cap \Omega_T] \leq c_0\,\omega^{\frac{2\kappa_1}{N+2\kappa_1}}\,\kappa_N\, R_0^{N+2} \; .$$

Consider inequalities $(5.1)^{-}$ for $(v - k)^{-}$; $k \leq \mu^{-} + \frac{\omega}{2^{s+2}}$, and apply lemma 3.1. It gives the following alternative. Either

(i) $$H = \operatorname*{ess\,sup}_{Q'_{R_0}} \left(v - \left(\mu^{-} + \frac{\omega}{2^{s+2}}\right)\right)^{-} \leq R_0^{\frac{N\kappa}{2}} \; ,$$

or

(ii) $$\operatorname{meas} Q^{-}_{\frac{R_0}{2}}\left(\mu^{-} + \frac{\omega}{2^{s+2}} - \frac{1}{2}H\right) = 0 \; .$$

If (i) occurs then

$$\operatorname*{ess\,inf}_{Q'_{R_0}\cap\Omega_T} v \geq \mu^{-} + \frac{\omega}{2^{s+2}} - R_0^{\frac{N\kappa}{2}} \; .$$

If (ii) is valid then

$$\operatorname*{ess\,inf}_{Q^{\omega}_{\frac{R_0}{2}} \cap \Omega_T} v > \mu^- + \frac{\omega}{2^{s+2}} - \frac{1}{2}\,\frac{\omega}{2^{s+2}}$$

$$= \mu^- + \frac{\omega}{2^{s+3}} \; .$$

Hence in either case

$$\operatorname*{osc}_{Q^{\omega}_{\frac{R_0}{2}} \cap \Omega_T} v < \omega - \frac{\omega}{2^{s+3}} \; .$$

The lemma is proved.

<u>Lemma 5.3</u>: Let $x_0 \in \operatorname{int} K \subset \Omega$ and R_0 so small that $B(R_0) \subset K$. There exists a pair of sequences $\{R_n\} \searrow 0$, $\{M_n\} \searrow 0$ such that

$$\operatorname*{osc}_{Q^{M_n}_{R_n} \cap \Omega_T} v \le M_n \qquad n = 1, 2, \ldots \; ,$$

The sequences $\{R_n\}$ and $\{M_n\}$ depend only upon the data and the modulus of continuity $\hat{\omega}_K(\cdot)$ of $x \to v_0(x)$ in K.

<u>Proof of lemma 5.3</u>: For $R_1 \le R_0$ define

$$M_1 = \max\{2M \; ; \; 2^{s+1}\,\hat{\omega}_K(R_1)\} \ge 2M \; ,$$

and select R_1 so small that

$$R_1^{\frac{M_1}{2}} \cdot \left(\frac{2M}{2^{s+3}}\right) \le \left(\frac{M_1}{2^{s+3}}\right) \; .$$

Then construct inductively the sequences $\{R_n\}$ and $\{M_n\}$ as follows:

$$M_{n+1} = \max\left\{M_n - \frac{M_n}{2^{s+3}}\ ;\ 2^{s+1}\,\omega_K(R_n)\right\}\ ;\ n = 1, 2, \dots$$

$$R_{n+1} = \min\left\{\left(\frac{M_n}{2^{s+3}}\right)^{\frac{2}{N\kappa}}\ ;\ \frac{R_n}{2}\right\} .$$

It is immediate to verify that $\{M_n\} \searrow 0$ and $\{R_n\} \searrow 0$. By virtue of Lemma 2.5, the conclusion of Lemma 5.3 holds true for $n = 1$. Suppose it holds for n and let us show that it holds for $n + 1$. By assumption $\operatorname*{osc}_{Q_{R_n}^{M_n}\cap\Omega_T} v < M_n$ and $R_n^{N\kappa/2} < \frac{M_n}{2^{s+3}}$. Hence by Lemma 5.2

$$\operatorname*{osc}_{Q_{\frac{R_n}{2}}^{M_n}\cap\Omega_T} v \le M_{n+1} .$$

To conclude the proof observe that

$$Q_{R_{n+1}}^{M_{n+1}} \cap \Omega_T \subset Q_{\frac{R_n}{2}}^{M_n} \cap \Omega_T .$$

<u>Proof of Theorem 5.1</u>: It is an immediate consequence of Lemma 5.3.

<u>Corollary</u>: Let Ω' be an open set contained in Ω and assume that $x \to v_0(x)$ is continuous on Ω' with modulus of continuity $\omega_K(\cdot)$ uniform on every compact $K \subset \Omega'$ then v is continuous on $K \times [0,T]$ and

$$|v(x_1,t_1) - v(x_2,t_2)| \le \omega_K(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

$\forall (x_i, t_i) \in K \times [0,T]$ $i = 1,2$. The continuous non-decreasing function $\omega_K(\cdot)$, $\omega_K(0) = 0$ depends only upon the data and $\omega_K(\cdot)$.

[B] The case of variational boundary data.

Consider formally the problem

$$(5.6)\qquad \begin{cases} \frac{\partial}{\partial t} \beta(v) - \operatorname{div} \vec{a}(x,t,v,\nabla_x v) + b(x,t,v,\nabla_x v) = 0 \\ \vec{a}(x,t,v,\nabla_x v) \cdot \vec{n}_{S_T}(x,t) = g(x,t,v) \quad \text{on } S_T \\ v(x,0) = v_0(x), \quad v_0(x) \neq 0 \quad \text{a.e. in } \Omega, \end{cases}$$

where $\vec{n}_{S_T} = (n_{x_1}, n_{x_2}, \ldots, n_{x_n})$ denotes the outer unit normal to S_T. We assume that $\vec{a}(x,t,v,p)$ and $b(x,t,v,p)$ satisfy $[A_1] - [A_2]$ and that $v_0 \in L_2(\Omega)$. On the boundary data $g(x,t,v)$ we assume that

[G] g is continuous over $S_T \times \mathbb{R}$ and admits an extension $g(x,t,v)$ over Ω_T such that

$$\left\| \frac{\partial}{\partial v} g(x,t,v) \quad , \quad \frac{\partial}{\partial x} g(x,t,v) \right\|_{\infty, \Omega_T} < C < \infty$$

for some positive constant C.

Essentially we are imposing on g a growth at most linear with respect to v.

By a weak solution of (5.6) we mean a function $v \in W_2^{1,1}(\Omega_T)$ satisfying

$$(5.7)\quad - \int_\Omega \beta(x,\tau)\,[v \gtrless 0]\varphi \, dx \Big|_{t_0}^{t} + \int_{t_0}^{t} \int_\Omega \beta(x,\tau)\,[v \gtrless 0] \frac{\partial}{\partial t} \varphi \, dx d\tau +$$

$$\int_{t_1}^{t_2}\int [illegible] \frac{\partial}{\partial t} v\,\varphi + a(x,t,v,\nabla_x v)\cdot\nabla_x\varphi + b(x,t,v,\nabla_x v)\varphi\,dx\,dt =$$

$$\int_{t_1}^{t_2}\int [illegible]\; a(x,t,v)\varphi\,d\sigma\,dt\;,$$

$\forall\varphi \in W_2^{1,1}(\Omega_T)$ and any interval $[t_1,t_2] \subset [0,T]$, and $v(x,0) = v_0(x)$ in the sense of the traces over [illegible].

In particular (5.7) holds for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$, hence a weak solution of (5.6) is also a weak solution of (1.6). Therefore by the results of section [illegible] v is continuous in Ω_T. Moreover if v_0 is continuous in [illegible] then v is continuous also at $t = 0$.

The aim of this section is to prove that the continuity of v can be extended to S_T and that if v_0 is continuous in $\bar{\Omega}$, then v is continuous in $\bar{\Omega}_T$. In a precise way we want to prove the following theorem.

<u>Theorem [illegible]</u>: Assume that [illegible] is a C^1 manifold in $\mathbb{R}^{N-1}$, suppose that [illegible] holds, and let $v \in W_2^{1,1}(\Omega_T)$ be a weak solution of (5.6) such that

$$[illegible] \le [illegible]\;.$$

For every [illegible] there exists a continuous non-decreasing function $[illegible] : \mathbb{R}^{+} \to \mathbb{R}^{+}$, [illegible]$(0)$ [illegible] such that

$$|v(x_1,t_1) - v(x_2,t_2)| \le [illegible](|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})\;,$$

for all $(x_i,t_i) \in \bar{\Omega} \times [illegible], i = 1,2$.

Moreover if v_0 is continuous over all $\bar{\Omega}$, then there exists $s \to \omega_0(s) : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_0(0) = 0$ continuous and non decreasing such that

$$|v(x_1,t_1) - v(x_2,t_2)| \leq \omega_0(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for all $(x_i,t_i) \in \bar{\Omega}_T$, $i=1,2$.

The functions $\omega_\sigma(\cdot)$ can be determined in dependence of the data and the positive number σ, and $\omega_0(\cdot)$ can be determined only in terms of the data and the modulus of continuity of v_0.

The proof of the theorem is essentially the same as the proof of interior regularity and is based on the same arguments of sections 3.4. The difference is that instead of working on cylinders Q_R here we are dealing with cylindrical domains of the type

$$\Omega \cap B(R) \times [t_0 - R^2, t_0] \quad .$$

We bound ourselves to describe the differences occurring in the proof. Fix $x_0 \in \partial\Omega$ and consider the portion of the boundary $\partial\Omega$ given by

$$S_0 \equiv \partial\Omega \cap \{|x - x_0| < \bar{R}\} \quad , \quad \bar{R} > 0 \text{ given} \quad .$$

Our arguments being local in nature, we may assume without loss of generality that S_0 lies on the hyperplane $x_N = 0$. Indeed this can always be achieved by a local change of coordinates in identity (5.7) written for example for test functions $\varphi(\cdot,t)$ supported in a neighborhood of x_0, for $t \in [0,T]$.

[B]$_1$. Inequalities analogous to (2.7):

Let $(x_0,t_0) \in S_T$, $t_0 > 0$ and set

$$B_R = \{|x - x_0| < R\} \cap \Omega \quad , \quad R \leq \bar{R}$$

$$C_R = B_R \times [t_0 - R^2, t_0] \quad , \quad R^2 < t_0 \ .$$

Since $\partial\Omega$ around x_0 is a portion of the hyperplane $x_n = 0$ and $R \leq \bar{R}$, B_R is the half ball $\{|x - x_0| < R, x_N > 0\}$ and C_R is the half cylinder obtained by intersecting the lateral cylinder (LC) Q_R with Ω_T. Moreover notice that since $R^2 < t_0$, C_R does not intersect Ω at $t = 0$.

Our next task is to derive inequalities analogous to (2.7) over the domains C_R and

$$C_R(\sigma_1, \sigma_2) = B_{R - \sigma_1 R} \times [t_0 - (1 - \sigma_2)R^2, t_0] \quad ,$$

$$\sigma_1, \sigma_2 \in (0,1) \quad .$$

This is done by selecting in (5.7) test functions $\varphi = \pm(v - k)^{\pm} \zeta^2$, where $(x,t) \to \zeta^2(x,t)$ is chosen as in (a).

All the terms on the left hand side of (5.7) are treated as in the derivation of (2.7) except for the different domain of integration. We remark in this connection that $\zeta(x,t)$ vanishes on the parabolic boundary of Q_R, and <u>not</u> on the parabolic boundary of C_R.

We estimate the term involving an integration over $\partial\Omega$ on the right hand side of (5.7) by transforming it in a volume integral as follows

$$I = \int_{t_0 - R^2}^{t_0} \int_{\partial\Omega} g(x,\tau,v)(v - k)^{\pm} \zeta^2(x,\tau) d\sigma d\tau =$$

$$= \int_{t_0 - R^2}^{t_0} \int_{\partial B_R} g(x,\tau,v)(v - k)^{\pm} \zeta^2(x,\tau) d\sigma d\tau =$$

$$= \int_{C_R} \int \operatorname{div}[g(x,\tau,v)(v - k)^{\pm} \zeta^2(x,\tau)] dx d\tau \quad .$$

We expand the integrand, use hypothesis [G] and perform routine calculations involving the Cauchy inequality $ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$, $\varepsilon > 0$, to obtain the estimate

$$I \le \gamma_1(\varepsilon) \int_{C_R}\int (v-k)_{\pm}^{+2} |\nabla_x \zeta|^2 \, dx d\tau +$$

$$+ \gamma_2(\varepsilon) \int_{C_R}\int \chi[(v-k)^{\pm} > 0]\zeta^2(x,\tau)\, dx d\tau +$$

$$+ \varepsilon \int_{C_R}\int |\nabla_x (v-k)^{\pm}|^2 \, \zeta^2(x,\tau)\, dx d\tau \quad ,$$

where $\varepsilon > 0$ is arbitrary and γ_1, γ_2 are constants depending upon the data and ε.

These remarks prove that there exists constants γ and δ such that for all $k \in \mathbb{R}$ satisfying

$$\operatorname*{ess\,sup}_{C_R} (v-k)^{\pm} < \delta \quad , \tag{5.8}$$

we have the inequalities

$$\|(v-k)^{\pm}\|^2_{V_2^{1,0}(C_R(\sigma_1,\sigma_2))} \le \gamma[(\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}] \, \|(v-k)^{\pm}\|^2_{2,C_R} + \tag{5.9}$$

$$+ \gamma \left\{ \int_{t_0-R^2}^{t_0} [\text{meas}\, A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}} d\tau \right\}^{\frac{2}{r}(1+\kappa)} + \sup_{t \in [t_0-R^2,\, t_0]} \Phi^{\pm}_a(k, t_0-R^2, t)$$

where $\Phi^{\pm}_a(k,t_0-R^2,t)$ is defined as $\Phi^{\pm}(k,t_0-R^2,t)$ except for the different domain of integration. Inequalities (5.9) hold for all k satisfying (5.8) and all $\sigma_1, \sigma_2 \in (0,1)$.

We remark that the constants γ and δ in (5.9) might differ from the analogous constants in (2.7). This is due of course to the extra term involved in the boundary integral.

Lemma 2.1 remains unchanged and Lemma 2.2 now is stated as follows.

Lemma 5.4. Let $\theta, k \in \mathbb{R}^+$, $\mu \geq \underset{\Omega_R \times [t_0 - \theta R^2, t_0]}{\text{ess sup}} (v - k)^+$, and $\nu > 0$ such that [illegible]. Set

$$\psi(x,t) = \psi(v(x,t)) = \ln^+ \left[\frac{\mu}{\mu - (v - k)^+ + \nu} \right]$$

then there exists a constant C depending only upon the data such that for all $t \in [t_0 - \theta R^2, t_0]$

$$\int_{\Omega_{R-\sigma_1 R}} \psi^2(x,t)dx \leq \int_{\Omega_R} \psi^2(x,t_0 - \theta R^2)dx +$$

$$+ \frac{C}{\sigma_1^2} (1 + \ln \frac{\mu}{\nu})(1 + \frac{R^{N\kappa}}{\nu^2}) \text{meas } \Omega_R \quad .$$

As remarked after Lemma 2.2, also in the present situation an analogous result holds for $(v - k)^-$, $k > 0$.

$[B]_2$. Proceeding in the proof we see that Lemma 3.1 holds in the present situation for the domain C_R instead of for the cylinder Q_R. The only modification regards the proof of the recursion inequalities [I] - [II]. For these we used the embedding inequality (2.11) valid for functions of $\overset{\circ}{W}{}_2^{1,1}(Q_R)$. In our case $(v - k)^{\pm} \zeta^2(x,t)$ do not vanish on the lateral boundary of C_R, therefore we must use inequality (2.12), and observe that for the domain $\Omega_R \times [t_0 - \theta R^2, t_0]$ one can consider the constant in (2.12) as independent of R.

Finally the last modification occurs in Lemma 3.4 in the use of DeGiorgi's inequality (2.9). Now such an inequality holds also for convex domain, therefore (2.9) is valid with $B(R)$ replaced by Ω_R. The remainder of the proof stays unchanged. The first assertion of the theorem is proved. For the second part we consider domain C_R with $t_0 - R^2 < 0$ and over them carry on the arguments of Lemma 5.2 - 5.3 with the modifications indicated above.

[C] The case of homogeneous Dirichlet boundary data.

We let $v \in W_2^{1,1}(\Omega_T)$ be a weak solution of (1.6) which in addition satisfies

$$v\Big|_{S_T} = 0 \qquad (x,t) \in S_T \quad ,$$

in the sense of the traces over S_T. In this paragraph we investigate under what assumptions on $\partial\Omega$ the interior continuity of v can be extended up to the lateral boundary S_T of Ω_T. On $\partial\Omega$ assume the following:

(P) $\exists\theta^\star > 0$, $R_0 > 0$ such that $\forall x_0 \in \partial\Omega$ and every ball

$B(R)$ centered at x_0, $R \leq R_0$,

$$\mathrm{meas}[\Omega \cap B(R)] < (1 - \theta^\star)\mathrm{meas}\, B(R) \quad .$$

Theorem 5.3: Let $v \in W_2^{1,1}(\Omega_T)$ be a weak solution of (1.6) such that $\|v\|_{\infty,\Omega_T} \leq M < \infty$, and $v\Big|_{S_T} = 0$ in the sense of the traces. There exist $0 < \eta < 1$ and a constant L such that

$$|v(x,t)| \leq L(\mathrm{dist}[(x,t),\partial\Omega])^\eta \quad .$$

Moreover if $v(x,0) = v_0(x)$ in the sense of the traces over Ω and if $v_0 \in C(\bar{\Omega})$, $v_0\Big|_{\partial\Omega} = 0$, then there exists a continuous non decreasing function

$\omega(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega(0) = 0$ such that

$$|v(x_1,t_1) - v(x_2,t_2)| \le \omega(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for all $(x_i,t_i) \in \bar{\Omega}_T$ $i = 1,2$. The numbers ν and L depend uniquely upon the data, whereas $\omega(\cdot)$ can be determined in dependence of the data and the modulus of continuity of v_0 in $\bar{\Omega}$.

The theorem is a consequence of the following inequalities valid on every (LC) Q_R.

$$(5.10)\quad \|(v-k)^{\pm}\|^2_{V_2^{1,0}[C_R(\sigma_1,\sigma_2)]} \le \gamma[(\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}]\,\|(v-k)^{\pm}\|^2_{C_R} +$$

$$+ \gamma\left\{\int_{t_0-R^2}^{t_0} [\text{meas } A^{\pm}_{k,R}(\tau) \cap \Omega]^{\frac{r}{q}}\, d\tau\right\}^{\frac{2}{r}(1+\kappa)} + \sup_{t \in [t_0 - R^2, t_0]} \psi^{\pm}_a(k, t_0 - R^2, t) ,$$

for all $k \in \mathbb{R}$ such that

$$(5.11)\qquad \operatorname*{ess\,sup}_{Q_R \cap \Omega_T} (v-k)^{\pm} < \delta .$$

Here δ is the same number introduced in (2.6) and γ is the same constant appearing in (2.7). The definition of $\psi_a(k,t_0 - R^2, t)$ is obvious.

Inequalities (5.10) are derived from identity (2.1) upon the choice of $\varphi = \pm(v-k)^{\pm}\zeta^2$, where ζ is selected as in (a).

Notice that since $v \in \overset{\circ}{W}{}^{1,1}_2(\Omega_T)$ we have also $(v-k)^{\pm} \in \overset{\circ}{W}{}^{1,1}_2(\Omega_T)$ and therefore such a choice of φ in (2.1) is justified. We will use a simplified version of (5.10) obtained by imposing further restrictions on the levels k.

If we select $k > 0$ and write (5.10) for $(v-k)^+$ we see that $\psi^+_a(k,t_0 - R^2,t) = 0$; moreover by looking at v as extended to be zero on that part of Q_R that remains outside Ω_T, the domains of integration in (5.10) can be replaced by $Q_R(\sigma_1,\sigma_2)$ and Q_R respectively. Hence for $(v-k)^+$, $k > 0$ we are led to the inequalities

$$(5.10)^+ \quad |(v-k)^+|^2_{V_2^{1,0}(Q_R(\sigma_1,\sigma_2))} \le \gamma[(\sigma_1 R)^{-2} + (\sigma_2 R^2)^{-1}]\, \|(v-k)^+\|^2_{2,Q_R}$$

$$+ \gamma \left\{ \int_{t_0-R^2}^{t_0} [\text{meas } A^+_{k,R}(\tau)]^{\frac{r}{q}} \, d\tau \right\}^{\frac{2}{r}(1+\kappa)}$$

valid for all $\sigma_1,\sigma_2 \in (0,1)$, and all $k > 0$ satisfying (5.11).

The same argument applied to $(v-k)^-$, $k < 0$ leads to inequalities for $(v-k)^-$, $k < 0$ to which we will refer as $(5.10)^-$. We remark explicitly that in $(5.10)^+$ [$(5.10^-$ resp.] there is no restriction on the levels k other than $k > 0$ ($k < 0$ resp.) and satisfying (5.11).

Let $(x_0,t_0) \in S_T$ $t_0 > 0$ be fixed and let R_0 be so small that $(2R_0)^2 < t_0$, so that Q_R, $0 < R \le 2R_0$ are lateral cylinders (LC), with common "vertex" (x_0,t_0). Set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_{2R_0} \cap \Omega_T} v \quad , \quad \mu^- = \operatorname*{ess\,inf}_{Q_{2R_0} \cap \Omega_T} v \quad , \quad \omega = \operatorname*{ess\,osc}_{Q_{2R_0} \cap \Omega_T} v$$

and, without loss of generality suppose that

$$\mu^+ \ge |\mu^-| \quad .$$

However let $s \in \mathbb{N}$ be the smallest positive integer such that

$$\frac{2M}{2^s} < \delta \quad ,$$

and observe that $\mu^+ - \frac{\omega}{2^s} > 0$. We will employ $(5.10)^+$ for the levels $k = \mu^+ - \frac{\omega}{2^p}$, $p \geq s$, $p \in \mathbb{N}$, over the cylinders Q_R, $R \leq R_0$.

Lemma 5.5: For every $\theta_1 > 0$, there exists a positive integer p (depending on θ_1) such that either

$$\text{(i)} \quad \frac{\omega}{2^p} \leq R_0^{\frac{N\kappa}{2}} \,, \quad \text{or}$$

$$\text{(ii)} \quad \operatorname{meas}\{(x,t) \in Q_{R_0} \,|\, v(x,t) > \mu^+ - \frac{\omega}{2^p}\} \leq \theta_1 \, \kappa_N \, R_0^{N+2} \,.$$

The number p depends upon the data and θ_1 and it is independent of ω and R_0.

Proof of Lemma 5.5: The lemma is proved in exactly the same way as Lemma 3.4. We remark that the estimate

$$\operatorname{meas}\{B(R_0) \setminus A^+_{\mu^+ - \frac{\omega}{2^s}}(t)\} > \theta^* \, \kappa_N \, R_0^N$$

for all $t \in [t_0 - R_0^2, t_0]$, which in Lemma 3.4 was derived from De Giorgi's inequality, in the present situation is automatic since $\partial\Omega$ satisfies (P).

Lemma 5.6: There exists a number $\theta_1 > 0$ such that if

$$\operatorname{meas}\{(x,t) \in Q_{R_0} \,|\, v > \mu^+ - \frac{\omega}{2^p}\} \leq \theta_1 \, \kappa_N \, R_0^{N+2} \,,$$

then either

$$\text{(i)} \quad H \quad \operatorname*{ess\,sup}_{Q_{R_0}} \left(v - (\mu^+ - \frac{\omega}{2^p})\right)^+ \leq R_0^{\frac{N\kappa}{2}}$$

or

$$\text{(ii)}\quad \operatorname{meas}\{(x,t)\in Q_{\frac{R_0}{2}} \mid v(x,t) > \mu^+ - \frac{\omega}{2^p} + \frac{1}{2}H\} = 0 \quad .$$

The number θ_1 depends only upon the data and not upon ω or R_0.

Proof of Lemma 5.6: The proof is the same as for Lemma 3.1. In this case it is in fact simpler because the term $\phi_a^+(k, t_0 - R_0^2, t) = 0$. It is this last fact also that makes θ_1 independent of ω.

As a consequence of Lemmas 5.5 - 5.6 we observe the following result

Lemma 5.7: Consider the decreasing sequence of numbers $\left\{\frac{R_0}{2^n}\right\}$ and the family of coaxial nested cylinders $Q_{\frac{R_0}{2^n}}$ with common "vertex" (x_0, t_0).

There exists a positive integer $q \in \mathbb{N}$ such that either

$$\underset{Q_{\frac{R_0}{2^{n+2}}}\cap\Omega_T}{\operatorname{osc}} v \le 2^q \left(\frac{R_0}{2^n}\right)^{\frac{N\kappa}{2}}$$

or

$$\underset{Q_{\frac{R_0}{2^{n+2}}}\cap\Omega_T}{\operatorname{osc}} v \le \left(1 - \frac{1}{2^q}\right) \underset{Q_{\frac{R_0}{2^n}}\cap\Omega_T}{\operatorname{osc}} v \quad .$$

The first assertion of the theorem follows from Lemma 5.7. The fact that we have an estimate of Hölder type near S_T is a consequence of Lemma 5.7 above and Lemma 5.8 of [18] page 96-97.

The second part of the theorem is proved by estimating the oscillation of v in lateral cylinders Q_R with $t_0 - R^2 < 0$, in the same way as indicated in part [B]. We omit the details.

Remark: If $g(x,t) \equiv 0$ on $S_T' \subset S_T$ where S_T' is an open set in the relative topology of S_T, then the continuity can be extended up to any compact $K \subset S_T'$, compact in the relative topology of S_T.

6. Uniform approximations:

A common device in the theory of existence of weak solutions of (1.1) subject to some initial data and to variational or Dirichlet boundary conditions, consists in solving a sequence of regularized versions of (1.1) to obtain the solution as a limit in a suitable sense of a sequence of solutions of regularized problems. It is of interest in the applications to construct the solution as a limit in the topology of the uniform convergence on compacts of Ω_T. One such application can be found in [7]. In this section we indicate how this can be realized.

Let $v \in V_2^{1,0}(\Omega_T)$ satisfy identity (1.5) for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ such that $t \to \varphi(x,t)$ has compact support in $[0,T]$. Suppose that there exists sequences $\{w_n\}$ and $\{v_n\} \equiv \{\beta^{-1}(w_n)\}$ such that

(6.1) $\quad w_n, v_n \in W_2^{1,1}(\Omega_T)$

$v_n \to v$ strongly in $L_2(\Omega_T)$ and weakly in $V_2^{1,0}(\Omega_T)$

$w_n \to w$ weakly in $L_2(\Omega_T)$, $w \in \beta(v)$

$a_i(x,t,v_n,\nabla_x v_n)$, $b(x,t,v_n,\nabla_x v_n) \to$

$a(x,t,v,\nabla_x v)$, $b(x,t,v,\nabla_x v)$ weakly in $L_2(\Omega_T)$

(6.2) $\quad w_n$ and v_n satisfy the identity

$$\int_\Omega w_n(x,\tau)\varphi(x,\tau)\Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega \Big\{-w_n(x,\tau)\frac{\partial}{\partial t}\varphi(x,\tau) +$$

$$+ \Big[\frac{1}{n}\nabla_x w_n + \hat{a}(x,\tau,v_n,\nabla_x v_n)\Big]\cdot\nabla_x\varphi + b(x,t,v_n,\nabla_x v_n)\varphi\Big\}dx\,d\tau = 0$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and all intervals $[t_0,t] \subset (0,T]$.

Since $w_n \in \beta(v_n)$ in the sense of the graph, (6.2) is the weak formulation of

$$\frac{\partial}{\partial t}\beta(v_n) - \operatorname{div}\vec{a}(x,t,v_n,\nabla_x v_n) - \frac{1}{n}\Delta\beta(v_n) + \tag{6.3}$$

$$+ b(x,t,v_n,\nabla_x v_n) \ni 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T) \quad .$$

Remark: Because of the regularizing term $-n^{-1}\Delta\beta(v_n)$, the functions $(x,t) \to w_n(x,t) \in \beta(v_n)$ and $(x,t) \to v_n(x,t)$ are Hölder continuous over Ω_T with exponent depending upon the data and n, (see [18]).

Regularizations like (6.3) are of the type of Hopf vanishing viscosity, and were used in [2].

Furthermore we assume that the weak solution v_n of (6.3) can be obtained as a weak $W_2^{1,1}(\Omega_T)$-limit of weak solutions of

$$\frac{\partial}{\partial t}\beta_m(v_n^m) - \operatorname{div}\vec{a}(x,t,v_n^m,\nabla_x v_n^m) - \frac{1}{n}\Delta v_n^m + \tag{6.4}$$

$$+ b(x,t,v_n^m,\nabla_x v_n^m) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T)$$

where $\{\beta_m(\cdot)\}$ is a sequence of continuously differentiable regularizations of the graph $\beta(\cdot)$ such that

$$0 < \gamma_0 \le \beta_m'(s) \qquad \forall s \in \mathbb{R}$$

$$\beta_m'(s) < 1 \qquad |s| \ge \frac{1}{m} \quad .$$

Namely we assume that $\beta_m(v_n^m)$, $v_n^m \in W_2^{1,1}(\Omega_T)$ uniformly in m and that

(i) $v_n^m \to v_n$ strongly in $L_2(\Omega_T)$, weakly in $W_2^{1,1}(\Omega_T)$

(ii) $\beta_m(v_n^m) \to w_n \in \beta(v_n)$ strongly in $L_2(\Omega_T)$ and weakly in $W_2^{1,1}(\Omega_T)$

(iii) $a_i(x,t,v_n^m,\nabla_x v_n^m)$, $b(x,t,v_n^m,\nabla_x v_n^m) \to$

$a_i(x,t,v_n,\nabla_x v_n)$, $b(x,t,v_n,\nabla_x v_n)$ weakly in $L_2(\Omega_T)$.

This second approximation is introduced only for technical reasons in order to justify the calculations below.

Theorem 6.1. Assume that $\exists M < \infty$ such that

$$\forall n \in \mathbb{N} \qquad \|v_n\|_{\infty,\Omega_T} \le M \ .$$

Then the sequence $\{v_n\}$ is equicontinuous in Ω_T.

If $\forall n \in \mathbb{N}$ $v_n(x,0) = v_0(x) \in C(\Omega)$ in the sense of the traces over Ω, then $\{v_n\}$ is equicontinuous in $\Omega_T \cup \Omega(0)$.

If $\forall n \in \mathbb{N}$ $v_n\big|_{S_T} = 0$ and $v_n(x,0) = v_0(x) \in C(\bar\Omega)$ then $\{v_n\}$ is equicontinuous in $\bar\Omega_T$.

Finally assume that

(i) $v_n(x,0) = v_0(x) \in C(\bar\Omega)$ $\quad \forall n \in \mathbb{N}$

(ii) $\{\vec a(x,t,v_n,\nabla_x v_n) - \frac{1}{n}\nabla_x v_n\} \cdot \vec n_{S_T} = g(x,t,v_n)$

in the sense made precise in (5.7) .

(iii) $\partial\Omega$ is a C^1 manifold in $\mathbb{R}^{N-1}$.

(iv) g satisfies assumptions [G] of Theorem 5.1 .

Then the sequence $\{v_n\}$ is equicontinuous in $\bar\Omega_T$.

Proof of Theorem 6.1: In section 4 we remarked that the modulus of continuity of v in Ω_T is determined uniquely in dependence of M and the various constants appearing in (2.7) and Lemma 2.2. In view of this, to prove the theorem will be enough to show that the functions $(v_n - k)^{\pm}$ satisfy inequalities like (2.7) and Lemma 2.2, with constants independent of n.

Let $(x_0, t_0) \in \Omega_T$ and R so small that $Q_R \subset \Omega_T$. Let $\sigma_1, \sigma_2 \in (0,1)$, construct the cylinder $Q_R(\sigma_1, \sigma_2)$ and smooth cutoff functions $(x,t) \to \zeta(x,t)$ such that

(i) $\zeta(x,t) \equiv 1 \qquad (x,t) \in Q_R(\sigma_1,\sigma_2)$, $\operatorname{supp} \zeta \subset Q_R$,

(ii) $|\zeta_t| \le \dfrac{C}{\sigma_2} R^{-2}$; $|\nabla_x \zeta| \le \dfrac{C}{\sigma_1} R^{-1}$

(iii) $|\Delta\zeta| \le \dfrac{C}{\sigma_1^2} R^{-2}$.

It is easily checked that Lemma 2.2 carries over to the present situation with constants independent of n. We start from (6.2) choose a cutoff function $x \to \zeta(x)$ independent of t, and reproduce the same estimates in the proof of Lemma 2.2.

Next observe that by selecting $\varphi = (v_n - k)^+ \zeta^2$, $k > 0$ in (6.2) we obtain inequalities (2.7) for $(v_n - k)^+$ with $\Lambda_a^+(k, t_0 - R^2, t) = 0$. The calculations show that the constants γ and δ are independent of n (although they might differ from the analogous constants in (2.7)). The argument remains valid for $(v_n - k)^-$, $k < 0$. Hence we have to prove inequalities like (2.7) for $(v_n - k)^+$, $k < 0$ and $(v_n - k)^-$, $k > 0$.

For this write (6.4) in the weak form for test functions $\varphi = \pm(v_n^m - k)^{\pm} \zeta^2$. The term

$$\int_{t_0-R^2}^{t_0} \int \{\vec{a}(x,t,v_n^m,\nabla_x v_n^m)\nabla_x[(v_n^m - k)^{\pm} \zeta^2] -$$

$$- b(x,t,v_n^m,\nabla_x v_n^m)(v_n^m - k)^{\pm} \zeta^2\} dx d\tau$$

can be treated in exactly the same way as in the derivation of (2.7). Then we let $m \to \infty$ (the lower semicontinuity of $\nabla_x v_n^m$ in $L_2(Q_T)$ is employed) and observe that the constants involved are independent of n.

Next we estimate the two remaining terms

$$I_1 = \int_{t_0-R^2}^{t_0} \int_{\Omega} \frac{\partial}{\partial t} \beta_m(v_n^m)[\pm(v_n^m - k)^{\pm}]\, \zeta^2(x,t)\,dxdt$$

$$I_2 = \frac{1}{n} \int_{t_0-R^2}^{t_0} \int_{\Omega} \nabla_x \beta_m(v_n^m) \nabla_x [\pm(v_n^m - k)^{\pm}\, \zeta^2(x,\tau)]\,dxd\tau \ .$$

For I_1 we have (we drop the subscripts m and n for simplicity of notation).

$$I_1 = \int\int_{Q_R} \beta'(\pm(v - k)^{\pm} + k)[\pm(v - k)^{\pm}]\frac{\partial}{\partial t}(v - k)^{\pm}\, \zeta^2(x,\tau)\,dxd\tau =$$

$$= \int\int_{Q_R} \frac{\partial}{\partial t} \Lambda[(v - k)^{\pm}] \cdot \zeta^2(x,\tau)\,dxd\tau \ ,$$

where

$$\Lambda(s) = \int_0^s \beta'(k + \xi)\xi\, d\xi \ .$$

It follows that

$$I_1 \geq c_0 \int_{B(R-\sigma_1 R)} [(v - k)^{\pm}]^2\, \zeta^2(x,t) - \frac{1}{c_2 R^2} \int\int_{Q_R} \Lambda[\pm(v - k)^{\pm}]\,dxd\tau \ .$$

The integral I_2 is estimated as follows:

$$nI_2 \geq -\int_{Q_R}\int \beta(v)\nabla_x(v-k)^{\pm}\cdot\nabla_x\zeta^2\,dxd\tau\ -$$

$$-\int_{Q_R}\int \beta(v)(v-k)^{\pm}\,\Delta\zeta\,dxd\tau\quad .$$

From this, standard calculations and limiting processes it follows that there exist constants γ, δ independent of n such that

$$(7.1)\quad |(v_n-k)^{\pm}|^2_{V_2^{1,0}(Q_R(\sigma_1,\sigma_2))} \leq \gamma[(\sigma_1 R)^{-2}+(\sigma_2 R^2)^{-1}]\,\|(v_n-k)^{\pm}\|^2_{2,Q_R}\ +$$

$$+\ \gamma\left\{\int_{t_0-R^2}^{t_0}[\text{meas}\,A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}}\,d\tau\right\}^{\frac{2}{r}(1+\kappa)}+\ \phi^{\pm}(k,t_0-R^2,t_0)$$

provided that $\operatorname*{ess\,sup}_{Q_R}(v-k)^{\pm}<\delta$.

Here $\phi^{\pm}(\cdot,\cdot,\cdot)$ can be majorized by

$$(7.2)\qquad \phi^{\pm}\leq \frac{\text{const}}{\min[\sigma_1;\sigma_2]R^2}\int_{Q_R}\int\{(v_n-k)^{\pm}+\chi[(v_n-k)^{\pm}>0]\}dxd\tau\quad .$$

Moreover $\phi^{\pm}$ vanishes if (7.1) are written for $(v_n-k)^+$ $k>0$, or $(v_n-k)^-$, $k<0$.

The term $\phi^{\pm}$ is slightly different from the $\phi^{\pm}_a$ in (2.7). The only part of the proof of Theorem 1 where (2.7) has been employed with $\phi^{\pm}_a\neq 0$ is Lemma 3.1. In such a lemma we estimated $\phi^{\pm}_a$ as

$$\phi^{\pm}_a(k,t_0-R^2,t)\ \leq\ \frac{2\nu}{\sigma_2 R^2}\int_{Q_R}\int(v-k)^{\pm}\,dxd\tau\quad .$$

By following the various steps in the proof of Lemma 3.1 it is easily checked that the extra term

$$\frac{1}{\min[\sigma_1,\sigma_2]R^2} \int_{Q_R}\int \chi[(v_n - k)^{\pm} > 0]dxd\tau$$

in (7.2) does not affect the result. A few minor changes are necessary which are left to the reader.

For the continuity up to the boundary the same arguments of section 6 are valid in the present situation. The proof is complete.

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It is demonstrated that weak solutions of (1.1) in the introduction are continuous in their domain of definition. The continuity up to the boundary is also investigated.

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